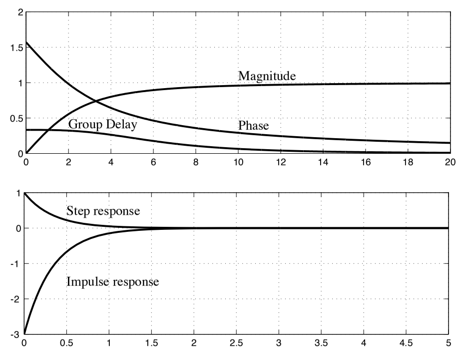
1. INTRODUCTION TO ANALOG FILTERS

1.1 a Transfer function (in Swedish: överföringsfunktion): $H(s) = \frac{s}{s+3}$

Frequency response (in Swedish: frekvenssvar): $H(j\omega) = \frac{j\omega}{j\omega + 3}$

Magnitude function (in Swedish: beloppsfunktion): $|H(j\omega)| = \frac{\omega}{\sqrt{\omega^2 + 9}}$

Phase response (in Swedish: fasvridning): $\Phi(\omega) = \arg\left\{\frac{j\omega}{j\omega + 3}\right\} = \frac{\pi}{2} - \operatorname{atan}\left(\frac{\omega}{3}\right)$



b) $\tau_g = -\frac{\partial \Phi(\omega)}{\partial \omega} = -\left(-\frac{1}{\left(\frac{\omega}{3}\right)^2 + 1}\frac{1}{3}\right) = \frac{3}{\omega^2 + 9}$

c) The inverse of $\frac{s}{s+3}X(s))$, where X(s) =1, is not included in the Laplace tables. Use the fact, see Table and Formulas for Analog and Digital Filters, that

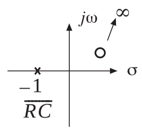
$\frac{d^k}{dt^k}x(t) \leftrightarrow s^k X(s) - s^{k-1}x(0) - \sum_{i=2}^{k} s^{k-i}\frac{d^{i-1}}{dt^{i-1}}x(t)\Big|_{t=0}$ for $k = 1$, i.e., $\frac{d}{dt}x(t) \leftrightarrow sX(s) - x(0)$. We get

from entry 11: $x(t) = L^{-1}\left\{\frac{1}{s+3}\right\} = e^{-3t}u(t)$ and $L^{-1}\left\{\frac{s}{s+3}1\right\} = \frac{d}{dt}L^{-1}\left\{\frac{1}{s+3}\right\} + x(0) =$

$= \frac{d}{dt}[e^{-3t}u(t)] + x(0) = -3e^{-3t}u(t) + \delta(t)$ Note there is an impulse $\delta(t)$ at $t = 0$ that is not visible

in the diagram. $s(t) = L^{-1}\left\{\frac{s}{s+3}\frac{1}{s}\right\} = e^{-3t}u(t)$

1.3 $H(s) = \frac{V_{out}(s)}{V_{in}(s)} = \frac{\frac{1}{sC}}{R + \frac{1}{sC}} = \frac{1}{sRC + 1} = \frac{\frac{1}{RC}}{s + \frac{1}{RC}}$



b) A pole at $s = -\frac{1}{RC}$ and a zero at $s = \infty$.

c) $H(j\omega) = \frac{V_{out}(j\omega)}{V_{in}(j\omega)} = \frac{\frac{1}{j\omega C}}{R + \frac{1}{j\omega C}} = \frac{1}{j\omega RC + 1}$

$|H(j\omega)| = \left|\frac{1}{j\omega RC + 1}\right| = \frac{1}{\sqrt{(\omega RC)^2 + 1}}$. It is a lowpass filter.

$\Phi(\omega) = \arg\left(\frac{1}{j\omega RC + 1}\right) = -\operatorname{atan}(\omega RC)$

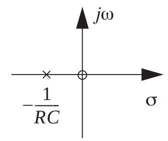
d) See Figures above.

e) $\tau_f(\omega) = \frac{-\Phi(\omega)}{\omega} = \frac{\operatorname{atan}(RC\omega)}{\omega}$ and $\tau_g(\omega) = \frac{-\partial\Phi(\omega)}{\partial\omega} = \frac{RC}{1 + (RC\omega)^2}$ See figures above.

f) $h(t) = \frac{1}{RC}e^{-\frac{t}{RC}}u(t)$ and $s(t) = L^{-1}\left\{\frac{a}{s+a}\frac{1}{s}\right\} = L^{-1}\left\{\frac{1}{s} - \frac{1}{s+a}\right\} = \left(1 - e^{-\frac{t}{RC}}\right)u(t)$ where $a = 1/RC$ See the figures above.

1.4 a)

a) $H(s) = \frac{V_{out}(s)}{V_{in}(s)} = \frac{R}{R + \frac{1}{sC}} = \frac{sRC}{sRC + 1} = \frac{s}{s + \frac{1}{RC}}$



A pole at $s = -\frac{1}{RC}$ and a zero at $s = 0$, i.e., a HP filter.

$H(j\omega) = \frac{V_{out}(j\omega)}{V_{in}(j\omega)} = \frac{R}{R + \frac{1}{j\omega C}} = \frac{j\omega RC}{j\omega RC + 1}$

$\Phi(\omega) = \frac{\pi}{2} - \operatorname{atan}(\omega RC)$ See the figures above.

b) $\tau_f = -\frac{\frac{\pi}{2} - \operatorname{atan}(\omega RC)}{\omega}$ and $\tau_g = -\frac{\partial\left(\frac{\pi}{2} - \operatorname{atan}(\omega RC)\right)}{\partial\omega} = \frac{RC}{(\omega RC)^2 + 1}$

c) $h(t)$ and $s(t)$. See the figures above.

1.5 a

b)

1.6 $A(\omega) = -20\log|H(j\omega_0)| = 1.25 \Rightarrow |H(j\omega_0)| = 10^{\frac{-1.25}{20}} = 10^{-0.0625} = 0.86596$

b) $A(\omega) = -20\log|H(j\omega_0)| = 40 \Rightarrow |H(j\omega_0)| = 10^{\frac{-40}{20}} = 10^{-2} = 0.01$

1.7 Gain = 0.75/0.5 = 1.5 and Phase = 5.2 – 0.4 = 4.8 rad

1.8 Transfer function: $H(s) = \frac{(s+3j)(s-3j)}{(s+3+2j)(s+3-2j)} = \frac{s^2 + 9}{s^2 + 6s + 13}$

b) Frequency response: $H(j\omega) = \frac{9 - \omega^2}{13 - \omega^2 + j6\omega}$

Magnitude function: $|H(j\omega)| = \frac{|9 - \omega^2|}{\sqrt{(13 - \omega^2)^2 + (6\omega)^2}}$

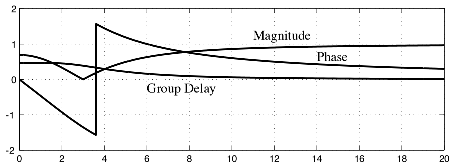
Phase response: $\Phi(\omega) = \arg\left\{\frac{9-\omega^2}{13-\omega^2+j6\omega}\right\} = \arg\{9-\omega^2\} - \arg\{13-\omega^2+j6\omega\}$

$$\Phi(\omega) = \begin{cases} -\text{atan}\left(\frac{6\omega}{13-\omega^2}\right), \omega < 3 \\ \pi - \text{atan}\left(\frac{6\omega}{13-\omega^2}\right), \omega > 3 \end{cases}$$

Group delay (in Swedish: grupplöptid)

$: \tau_g(\omega) = -\frac{\partial\Phi(\omega)}{\partial\omega} = \frac{(6(13-\omega^2)-6\omega(-2\omega))}{(13-\omega^2)^2\left(\left(\frac{6\omega}{13-\omega^2}\right)^2+1\right)} = \frac{6\omega^2+78}{\omega^4+10\omega^2+169}$

c)

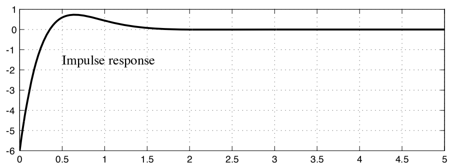


d) Impulse response (in Swedish: impulssvar)

$: H(s) = \frac{s^2+9}{s^2+6s+13} = 1 + \frac{-6s-4}{s^2+6s+13} = 1 + \frac{-3(2s)}{(s+3)^2+2^2} + \frac{-2(2)}{(s+3)^2+2^2}$

$L^{-1}\{H(s)\} = \delta(t) + (-3e^{3t}(2\cos((2t)-3\sin(2t))) - 2e^{3t}\sin(2t))u(t)$ and

$h(t) = \delta(t) + e^{-3t}[7\sin(2t) - 6\cos(2t)]u(t)$ or use the function
[h, dirac0, t_axis] = PZ_2_IMPULSE_RESPONSE_S(G, Z, P, t_axis)



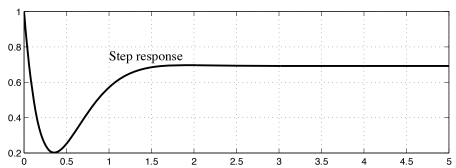
There is an impulse at $t = 0$ that is not visible in the diagram.

d) Step response (stegsvar): $s(t) = \int_0^t h(t)dt = \int_0^t (\delta(t) + e^{-3t}[7\sin(2t) - 6\cos(2t)]u(t))dt =$

$= u(t) + 7\left(\frac{1}{13}[-2\cos((2t)-3\sin(2t))] + 2\right)e^{-3t} - 6\frac{1}{13}[-3\cos(2t) + 2\sin(2t) + 3]$

$s(t) = \frac{9}{13}u(t) + e^{-3t}\left[\frac{4}{13}\cos(2t) - \frac{33}{13}\sin(2t)\right]u(t)$ Note that $s(0) = 1$

or use the function **[s_of_t, t_axis] = PZ_2_STEP_RESPONSE_S(G, Z, P, t_axis)**



1.9 a) $\tau_f = -\frac{\Phi}{\omega}$ and $\tau_g = -\frac{\partial\Phi}{\partial\omega}$

b) In order to retain the wave form of the input signal. Constant group delay implies that the envelop, i.e., the wave form is retained. Of course, the wave form is affected if significant frequency components are attenuated. It is important to have small variations in the group delay in application where the interesting information is in the wave form, e.g., EEG, ECG and images.

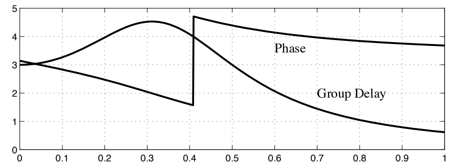
c) $H(j\omega) = \frac{j\omega L}{R + \frac{1}{j\omega C} + j\omega L} = \frac{-CL\omega^2}{1 - CL\omega^2 + jCR\omega}$ and $\Phi(\omega) = \arg(-CL\omega^2) - \arg\left(\frac{\omega RC}{1-CL\omega^2}\right)$

$\Phi(\omega) = \pi - \text{atan}\left(\frac{\omega RC}{1-CL\omega^2}\right)$

$\tau_f = -\frac{\pi - \text{atan}\left(\frac{\omega RC}{1-CL\omega^2}\right)}{\omega}$

$\tau_g = -\frac{d}{d\omega}\pi - \text{atan}\left(\frac{\omega RC}{1-CL\omega^2}\right) = \frac{RC(1-CL\omega^2) - 2\omega^2 RLC^2}{\left(\left(\frac{\omega RC}{1-CL\omega^2}\right)^2+1\right)(1-CL\omega^2)} = \frac{RC(\omega^2 LC+1)}{(\omega RC)^2 + (LC\omega^2-1)^2}$

We get with $R = 1\ \Omega$, $L = 2$ H, and $C = 3$ F.



1.10

1.11

1.12 a) $Area = \int_0^\infty \tau_g(\omega)d\omega = -\int_0^\infty \frac{\partial}{\partial\omega}\Phi(\omega)d\omega = \Phi(0) - \Phi(\infty) = n\frac{\pi}{2}$

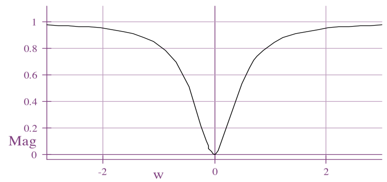
b) $\Phi(0)$ is always a multiple of π and $\Phi(\infty)$ is always a multiple of $\pi/2$. The area is therefore a multiple of $\pi/2$ and depend on the poles and zeros. Each pole contribute with the area $\pi/2$ and it is mainly concentrated in the frequency range around the pole frequency, ω_p. If the pole is close to the $j\omega$-axis, the main part of the area is concentrated at the pole frequency, while for poles far way for the $j\omega$-axis, that area in distributed over a larger frequency range. Zeros in the LHP contribute in the same way to the area, but zeros in the RHP contribute with a negative area. Zeros on the $j\omega$-axis do not contribute the any area.

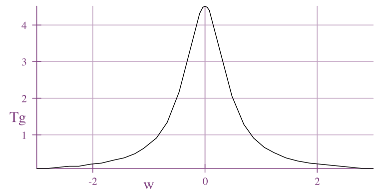
1.13) $H(j\omega) = \frac{-\omega^2}{-\omega^2 + 0.2 + j0.9\omega}$ and $|H(j\omega)| = \frac{\omega^2}{\sqrt{(0.2-\omega^2)^2 + 0.81\omega^2}}$

$\Phi(\omega) = \arg\{H(j\omega)\} = \arg(-\omega^2) - \text{atan}\left(\frac{0.9\omega}{0.2-\omega^2}\right) = \pi - \text{atan}\left(\frac{0.9\omega}{0.2-\omega^2}\right)$

Group delay: $\tau_g = -\frac{\partial\Phi}{\partial\omega} = \frac{0.9(\omega^2+0.2)}{\left(\left(\frac{0.9\omega}{0.2-\omega^2}\right)^2+1\right)(0.2-\omega^2)^2} = \frac{0.9(\omega^2+0.2)}{(\omega^2+0.16)(\omega^2+0.25)}$

b)



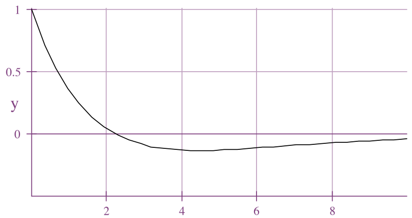


c) The step response is obtained for the input: $X(s) = \frac{1}{s}$

$$Y(s) = H(s)X(s) = \frac{s^2}{(s+0.4)(s+0.5)}\frac{1}{s} = \frac{-4}{(s+0.4)} + \frac{5}{(s+0.5)}$$

$$=> s(t) = (-4e^{-0.4t} + 5e^{-0.5t})u(t)$$

The impulse response is $h(t) = \frac{d}{dt}s(t) = (0.16e^{-0.4t} - 0.25e^{-0.5t})u(t)$

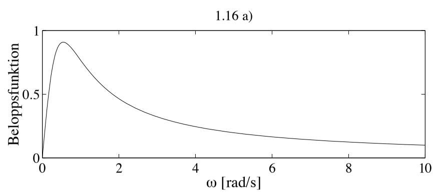


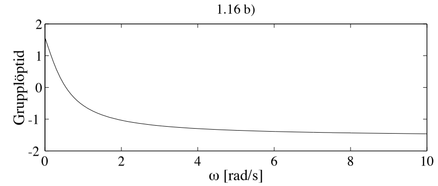
1.14 a) $|H(j\omega)| = \left|\frac{j\omega}{0.3-\omega^2+j1.1\omega}\right| = \frac{\omega}{\sqrt{(0.3-\omega^2)^2+(1.1\omega)^2}}$

$\boldsymbol{\Phi(\omega)} = \arg\{H(j\omega)\} = \arg(j\omega) \; -\text{atan}\left(\frac{1.1\omega}{0.3-\omega^2}\right) = \pi/2 - \text{atan}\left(\frac{1.1\omega}{0.3-\omega^2}\right)$

Group delay: $\tau_g = -\frac{\partial\Phi}{\partial\omega} = 1.1\frac{(0.3+1.2\omega^2)}{\omega^4+0.61\omega^2+0.09}$

b)



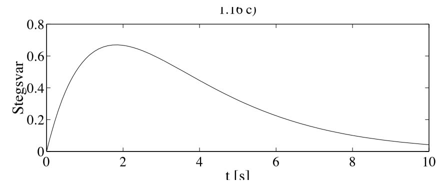


c) The step response is obtained with: X(s) = 1/s.

$$s(t) = L^{-1}\left\{\frac{s}{(s+0.5)(s+0.6)}\frac{1}{s} = \frac{10}{s+0.5} - \frac{10}{s+0.6}\right\} = 10(e^{-0.5t} - e^{-0.6t})u(t)$$

The impulse response is

$$h(t) = \frac{d}{dt}s(t) = (-5e^{-0.5t} + 6e^{-0.6t})u(t)$$

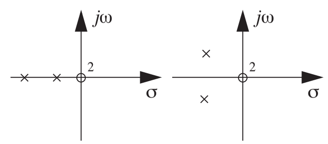


1.15

1.16) $H(s) = \frac{V_{out}(s)}{V_{in}(s)} = \frac{sL}{R+\frac{1}{sC}+sL} = \frac{s^2LC}{s^2LC+sRC+1} = \frac{s^2}{s^2+s\frac{R}{L}+\frac{1}{LC}}$

b) A complex conjugate pole pair at $s = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$ and two zeros at $s = 0$,

$G = 1$, i.e., a HP filter. The pole-zero configuration in the s-plane is, if $CR^2 < 4L$, according the figure to the right, or as in the left figure, i.e., two real poles and two zeros at $s = 0$ if $CR^2 > 4L$.



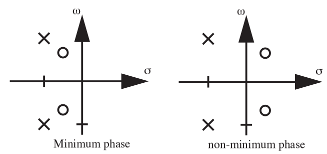
c)

$$H(j\omega) = \frac{V_{out}(j\omega)}{V_{in}(j\omega)} = \frac{j\omega L}{R+\frac{1}{j\omega C}+j\omega L} = \frac{\omega^2LC}{\omega^2LC - j\omega RC - 1}$$

d) **Figures**

1.17) A filter that has the smallest phase, i.e., all zeros are in the left half of the s-plane or on the $j\omega$-axis. For example, two filter with the transfer functions $H_{minfas}(s) = (s+a)/(s+b)$ and $H_{maxfas}(s) = (s-a)/(s+b)$ where $a > 0$.

b)



1.18 a) Minimum-phase => all pole and zeros in the LHP or on the $j\omega$-axis => 2, 4 and 6.

b) Allpass => every pole have a zero symmetrically on the other side of the $j\omega$-axis => 1 and 3.

c) Stable => no poles in the RHP or on the $j\omega$-axis => 1, 2, 4 and 6.

1.19 a) $H(s) = \frac{s^2 - 2s + 2}{s^2 + 2s + 2}$

b) $|H(j\omega)| = 1$

c) $\arg\{H(j\omega)\} = \arg(-\omega^2 - 2j\omega + 2) - \arg(-\omega^2 + 2j\omega + 2) =$

$$= \operatorname{atan}\left(\frac{-2\omega}{2-\omega^2}\right) - \operatorname{atan}\left(\frac{2\omega}{2-\omega^2}\right) = -2\operatorname{atan}\left(\frac{2\omega}{2-\omega^2}\right)$$

d) Allpass filter

e) Correct the group delay of a filter with the desired magnitude function but with too large variation in the group delay.

1.20