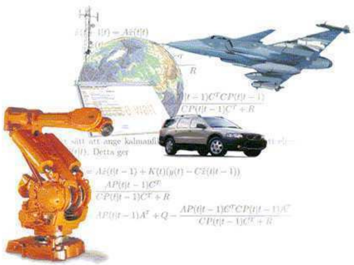
Robust Multivariable Control

Lecture 2



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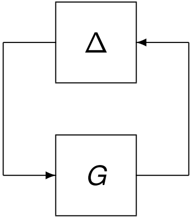
Today's topics

- Norms
- Representation of dynamic systems
- Lyapunov equations
- Gramians
- Balancing
- Model reduction
- The H_2 norm
- The H_∞ norm

Small gain theorem



If G is stable and $\|G\| < 1$ then the closed loop system is stable if $\|\Delta\| \leq 1$.

Which norm should we use?

Norms

p-norms for vectors:

$$\|x\|_p = \left(\sum_i |x_i|^p \right)^{1/p}$$

Most common:

- $p = 1$: sum of the absolute values of the elements;
- $p = 2$: Euclidian norm of the vector x;
- $p = \infty$: the maximum of the absolute values.

Induced norms

$$\|A\| = \sup_{x \neq 0} \frac{\|Ax\|_2}{\|x\|_2} = \bar{\sigma}(A)$$

maximum singular value.

Singular values

Singular value decomposition

$$A = U\Sigma V^*, \qquad U^*U = I, \quad V^*V = I$$

where

$$\Sigma = \begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_n \end{bmatrix} \in \mathbb{R}^{n \times n}$$

assuming $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n \geq 0$.

$$(A^*A)V = (U\Sigma V^*)^* U\Sigma V^* V = V\Sigma U^* U\Sigma = V\Sigma^2,$$

which means that Σ^2 are the eigenvalues of A^*A.

Numerically, this is not a good method to compute Σ.

Singular values

Also,

$$\begin{bmatrix} 0 & A \\ A^* & 0 \end{bmatrix} \begin{bmatrix} U & U \\ V & -V \end{bmatrix} = \begin{bmatrix} 0 & U\Sigma V^* \\ V\Sigma U^* & 0 \end{bmatrix} \begin{bmatrix} U & U \\ V & -V \end{bmatrix}$$
$$= \begin{bmatrix} U\Sigma & -U\Sigma \\ V\Sigma & V\Sigma \end{bmatrix} = \begin{bmatrix} U & U \\ V & -V \end{bmatrix} \begin{bmatrix} \Sigma & 0 \\ 0 & -\Sigma \end{bmatrix}$$

and, consequently, $\pm\Sigma$ are the eigenvalues of

$$\begin{bmatrix} 0 & A \\ A^* & 0 \end{bmatrix}$$

Singular values and LMIs

We can also use a Linear Matrix Inequality (LMI) to find the maximum singular value. Find the minimum γ for which

$$\begin{bmatrix} \gamma I & A \\ A^* & \gamma I \end{bmatrix} \succeq 0.$$

Here $\succeq 0$ means positive semidefinite (all eigenvalues ≥ 0).

Also,

$$\min_{\gamma}\{\gamma : \gamma I - A^* A \succeq 0\} = \bar{\sigma}^2(A)$$

Representation of dynamic systems

State-space representation

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

Transfer function

$$G(s) = C(sI - A)^{-1}B + D$$

Sometimes we write this as

$$\begin{bmatrix} \dot{x} \\ y \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix}$$

or

$$G = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

Serial and parallel connection

$$G_1 = \left[\begin{array}{c|c} A_1 & B_1 \\ \hline C_1 & D_1 \end{array}\right] \qquad G_2 = \left[\begin{array}{c|c} A_2 & B_2 \\ \hline C_2 & D_2 \end{array}\right]$$

Serial connection:

$$G_1 G_2 = \left[\begin{array}{cc|c} A_1 & B_1 C_2 & B_1 D_2 \\ 0 & A_2 & B_2 \\ \hline C_1 & D_1 C_2 & D_1 D_2 \end{array}\right]$$

Parallel connection:

$$G_1 + G_2 = \left[\begin{array}{cc|c} A_1 & 0 & B_1 \\ 0 & A_2 & B_2 \\ \hline C_1 & C_2 & D_1 + D_2 \end{array}\right]$$

Lyapunov equaions

The linear equation

$$A^T X + XA + Q = 0$$

is a *Lyapunov equation*.

It has a unique solution if A och $-A$ have no common eigenvalues; for instance if A has all its eigenvalues in the left-hand plane.

In Matlab we can solve the equation using `X = lyap (A', Q);`

Observability gramian L_o

The *observability gramian* is defined by

$$A^T L_o + L_o A + C^T C = 0.$$

Regard the following system ($u = 0$):

$$\begin{cases} \dot{x} = Ax \\ y = Cx \end{cases}$$

Introduce $V_o = x^T L_o x$ and take its time derivative

$$\begin{aligned} \dot{V}_o &= \dot{x}^T L_o x + x^T L_o \dot{x} = x^T A^T L_o x + x^T L_o A x \\ &= x^T \underbrace{\left(A^T L_o + L_o A\right)}_{-C^T C} x = -x^T C^T C x = -y^T y = -\left|y\right|^2 \end{aligned}$$

Observability gramian L_o

Integrate $\dot{V}_o$

$$\int_0^T \dot{V}_o dt = V_o(x(T)) - V_o(x(0)) = -\int_0^T |y(t)|^2 \, dt.$$

If we assume that the system is stable and $T \to \infty$ then $x(T)$ approaches zero.

Thus

$$V_o(x(0)) = x^T(0) L_o x(0) = \int_0^\infty |y(t)|^2 \, dt = \|y\|_2^2$$

The gramian, L_o, includes information about how much a certain state becomes “visible” (as energy) in the output signal.

Controllabilty gramian L_c

The *controllability gramian* is defined by

$$AL_c + L_cA^T + BB^T = 0.$$

Introduce $V_c(x) = x^T L_c^{-1} x$. This is a measure of the smallest energy needed in the input to obtain a certain state x.

As before, take the derivative of V_c with $\dot{x} = Ax + Bu$:

$$\begin{aligned}\dot{V}_c(x) &= \dot{x}^T L_c^{-1} x + x^T L_c^{-1} \dot{x} = (Ax+Bu)^T L_c^{-1} x + x^T L_c^{-1}(Ax+Bu) \\ &= u^T B^T L_c^{-1} x + x^T L_c^{-1} Bu + x^T \underbrace{(A^T L_c^{-1} + L_c^{-1} A)}_{-L_c^{-1} BB^T L_c^{-1}} x \\ &= -(B^T L_c^{-1} x - u)^T (B^T L_c^{-1} x - u) + u^T u \le u^T u\end{aligned}$$

Controllability gramian L_c

The control law $u_{\text{opt}} = B^T L_c^{-1} x$ gives the smallest control energy.

The energy needed for reaching a certain state, starting in $x(-\infty) = 0$, is given by

$$\|u\|_2^2 = V_c(x) + \|u - B^T L_c^{-1} x\|_2^2$$

Thus, $\|u\|_2^2 \geq V_c(x) = x^T L_c^{-1} x$, with equality when $u = u_{\text{opt}}$.

Balancing

The representation of a system is not unique.

$$G = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

Using a similarity transformation, it can also be written as

$$G = \left[\begin{array}{c|c} \hat{A} & \hat{B} \\ \hline \hat{C} & \hat{D} \end{array}\right] = \left[\begin{array}{c|c} TAT^{-1} & TB \\ \hline CT^{-1} & D \end{array}\right]$$

The transformation also affects the gramians

$$\hat{L}_o = T^{-T} L_o T^{-1}$$

and

$$\hat{L}_c = T L_c T^T$$

Balancing

Choose T in a "clever" way such that

$$\hat{L}_o = \hat{L}_c = \Sigma = \begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_n \end{bmatrix}$$

Then

$$\sigma_i^2 = \lambda_i(\hat{L}_o \hat{L}_c) = \lambda_i(L_o L_c).$$

To obtain this factor L_o och L_c, for instance using Cholesky decomposition: $L_o = U_o^T U_o$ and $L_c = U_c^T U_c$. Then compute $U \Sigma V^T = U_o U_c^T$ using singular value decomposition, where $UU^T = VV^T = I$. The transformation matrix is given by $T = \Sigma^{-\frac{1}{2}} U^T U_o$.

An example of balancing

$$G(s) = \frac{1}{(s+1)(s+5)} = \left[\begin{array}{cc|c} -5 & 0 & 1 \\ 1 & -1 & 0 \\ \hline 0 & 1 & 0 \end{array}\right] = \left[\begin{array}{cc|c} -0.5946 & 1.336 & -0.3656 \\ -1.336 & -5.405 & -0.3656 \\ \hline -0.3656 & 0.3656 & 0 \end{array}\right]$$

with $\Sigma = \mathrm{diag}\,[\sigma_1, \sigma_2] = \mathrm{diag}\,[0.1124, 0.0124]$.

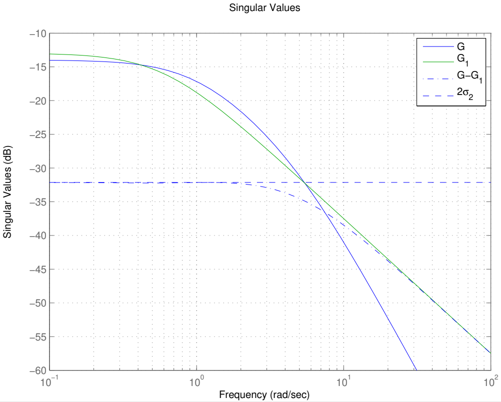
In Matlab this is done by

```
g = tf(1,[1 1])*tf(1,[1 5]); [gb,sig] = balreal(g);
```

Here σ_2 is small compared to σ_1. If we remove the corresponding state, x_2, the H_∞ error is limited by $2\sigma_2$:

$$\|G - \hat{G}_1\|_\infty \leq 2\sigma_2, \qquad \hat{G}_1(s) = \left[\begin{array}{c|c} -0.5946 & -0.3656 \\ \hline -0.3656 & 0 \end{array}\right]$$

An example of balancing



Model reduction

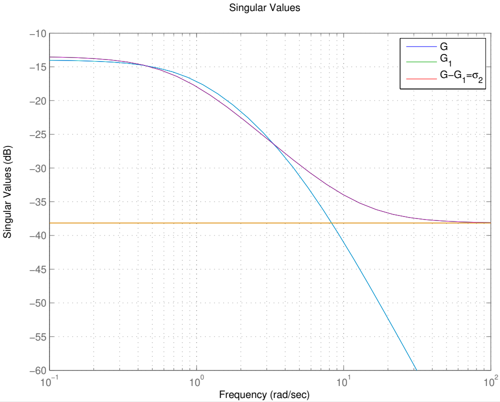
Model reduction can be done by truncating the states that have small Hankel singular values:

$$\|G_n - \hat{G}_r\|_\infty \leq 2 \sum_{i=r+1}^{n} \sigma_i$$

This upper bound can be halved by using a different scheme that also modifes the D matrix:

$$\sigma_{r+1} \leq \|G_n - \hat{G}_r\|_\infty \leq \sum_{i=r+1}^{n} \sigma_i$$

With modified D – optimal Hankel



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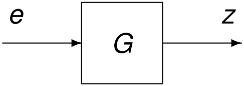
The H_2 norm



Let $e = \delta$ (dirac pulse) or white noise

$$\dot{x} = Ax + \underbrace{B\delta(t-0)}_{\Rightarrow \Delta x = B}$$

Energy in the output:

$$V(x) = \Delta x^T(0) L_o \Delta x(0) = B^T L_o B = \|G\|_2^2$$
$$= C L_c C^T$$

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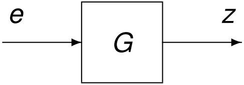
H_2 norm



In the MIMO case

$$\|G\|_2^2 = \operatorname{tr} B^T L_o B$$
$$= \operatorname{tr} C L_c C^T$$

The H_2 norm can be computed by solving a linear matrix equation.

The H_∞ norm

The H_∞ norm is a measure of the maximum gain of a stable system, G, over all frequencies, ω. That is to say

$$\|G\|_\infty = \max_\omega \bar{\sigma}\left(G(j\omega)\right)$$

L_∞ norm also applies to unstable systems.

H_∞ norm cannot be computed direct (as the H_2 norm), but we can test if $\|G\|_\infty < \gamma$.

H_∞ norm

$$G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = D + C(sI - A)^{-1}B$$

Assume that $\|G\|_\infty < \gamma$. If G is SISO then

$$\begin{aligned} \Phi(j\omega) &= \gamma^2 - \bar{\sigma}^2\left(G(j\omega)\right) \\ &= \gamma^2 - G(j\omega)^* G(j\omega) > 0, \end{aligned}$$

for all ω.

If $\gamma \le \|G\|_\infty$ then $\Phi(j\omega)$ has at least one zero for $\omega \in \mathbb{R}$.

H_∞ norm

$$\Phi(s) = \gamma^2 I - G^{\sim}(s)G(s) = \gamma^2 I - G(-s)^T G(s)$$

where

$$G^{\sim}(s) = \left[\begin{array}{c|c} -A^T & -C^T \\ \hline B^T & D^T \end{array}\right]$$

This gives

$$\Phi(s) = \left[\begin{array}{cc|c} A & 0 & B \\ -C^T C & -A^T & -C^T D \\ \hline -D^T C & -B^T & \gamma^2 I - D^T D \end{array}\right]$$

Requirement 1: $\gamma > \bar{\sigma}(D) = \sqrt{\lambda_{\max}(D^T D)}$.

If $\Phi(s)$ has a zero on the imaginary axis then $\Phi^{-1}(s)$ must have a pole on the imaginary axis.

Inverse of systems

We need to compute the inverse of a state space system.

The inverse of a system can be written as

$$G^{-1}(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]^{-1} = \left[\begin{array}{c|c} A - BD^{-1}C & -BD^{-1} \\ \hline D^{-1}C & D^{-1} \end{array}\right]$$

since ...

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Inverse of systems

$$
G^{-1}(s)G(s) = \left[\begin{array}{cc|c} A - BD^{-1}C & -BD^{-1}C & -BD^{-1}D \\ 0 & A & B \\ \hline D^{-1}C & D^{-1}C & D^{-1}D \end{array}\right]
$$

$$
= \left[\begin{array}{cc|c} I & I & 0 \\ 0 & I & 0 \\ \hline 0 & 0 & I \end{array}\right] \left[\begin{array}{cc|c} A - BD^{-1}C & -BD^{-1}C & -B \\ 0 & A & B \\ \hline D^{-1}C & D^{-1}C & I \end{array}\right] \left[\begin{array}{cc|c} I & -I & 0 \\ 0 & I & 0 \\ \hline 0 & 0 & I \end{array}\right]
$$

$$
= \left[\begin{array}{cc|c} A - BD^{-1}C & A - BD^{-1}C & 0 \\ 0 & A & B \\ \hline D^{-1}C & D^{-1}C & I \end{array}\right] \left[\begin{array}{cc|c} I & -I & 0 \\ 0 & I & 0 \\ \hline 0 & 0 & I \end{array}\right]
$$

$$
= \left[\begin{array}{cc|c} A - BD^{-1}C & 0 & 0 \\ 0 & A & B \\ \hline D^{-1}C & 0 & I \end{array}\right] \sim I
$$

H_∞ norm

The inverse of the system can be written as

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]^{-1} = \left[\begin{array}{c|c} A - BD^{-1}C & -BD^{-1} \\ \hline D^{-1}C & D^{-1} \end{array}\right]$$

Let H be the A matrix of the inverse of $\Phi(s)$:

$$H = \begin{bmatrix} A & 0 \\ -C^T C & -A^T \end{bmatrix} - \begin{bmatrix} B \\ -C^T D \end{bmatrix} \underbrace{\left(\gamma^2 I - D^T D\right)}_{R \succ 0}^{-1} \begin{bmatrix} -D^T C & -B^T \end{bmatrix}$$

$$= \begin{bmatrix} A + BR^{-1}D^T C & BR^{-1}B^T \\ -C^T C - C^T D R^{-1} D^T C & -A^T - C^T D R^{-1} B^T \end{bmatrix}$$

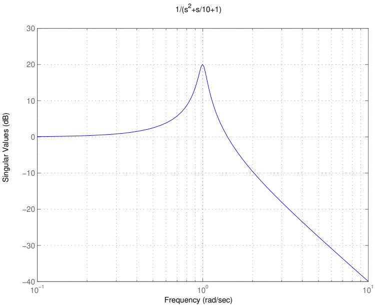
If H has eigenvalues on the imaginary axis then $\gamma \le \|G\|_\infty$.

If H has no eigenvalues on the imaginary axis then $\gamma > \|G\|_\infty$.

Example of the H_∞ norm

Let $G(s) = \frac{1}{1+0.1s+s^2}$:

γ	eig H
10	$\pm 1j, \pm 0.9950j$
10.1	$\pm 0.0066 \pm 0.9975j$
10.0125	$\pm \varepsilon \pm 0.9974j$



Induced norms

The H_∞ norm can also be described as an induced L_2 or ℓ_2 norm.

For a discrete-time system, if $\gamma > \|G\|_\infty$ then

$$\max_u \sum_{t=0}^{T} \left(y^T(t)y(t) - \gamma^2 u^T(t)u(t) \right) < 0$$

for the system

$$\begin{bmatrix} x(t+1) \\ y(t) \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}.$$

Induced norms

Introduce a Lyapunov function, $V(x) = x^T P x$, where $P = P^T \succ 0$ is positive definite:

$$
\begin{aligned}
&\sum_{t=0}^{T} \left(y^T(t)y(t) - \gamma^2 u^T(t)u(t) \right) \\
&= \sum_{t=0}^{T} \left(y^T(t)y(t) - \gamma^2 u^T(t)u(t) \right) + V(x(T+1)) - V(x(0)) + V(x(0)) - V(x(T+1)) \\
&= \sum_{t=0}^{T} \left(y^T(t)y(t) - \gamma^2 u^T(t)u(t) + V(x(t+1)) - V(x(t)) \right) + V(x(0)) - V(x(T+1)) \\
&= \sum_{t=0}^{T} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}^T \left(\begin{bmatrix} A^T P A - P & A^T P B \\ B^T P A & B^T P B - \gamma^2 I \end{bmatrix} + \begin{bmatrix} C^T \\ D^T \end{bmatrix} \begin{bmatrix} C & D \end{bmatrix} \right) \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \\
&\quad + V(x(0)) - V(x(T+1))
\end{aligned}
$$

LMIs

Assure that

$$\begin{bmatrix} A^T PA - P & A^T PB \\ B^T PA & B^T PB - \gamma^2 I \end{bmatrix} + \begin{bmatrix} C^T \\ D^T \end{bmatrix} \begin{bmatrix} C & D \end{bmatrix} \prec 0$$

or

$$\begin{bmatrix} A^T PA - P & A^T PB & C^T \\ B^T PA & B^T PB - \gamma^2 I & D^T \\ C & D & -I \end{bmatrix}$$

by choosing an appropriate matrix $P = P^T \succ 0$.

The smallest γ that we can find such a P for gives the upper bound of $\|G\|_\infty$.

This is a linear matrix inequality (LMI), which is a convex problem.

Schur complement

The condition

$$\begin{bmatrix} A & B \\ B^T & R \end{bmatrix} \prec 0$$

is equivalent to

$$\begin{bmatrix} A - BR^{-1}B^T & \\ & R \end{bmatrix} \prec 0$$

This follows from a congruence transformation

$$\begin{bmatrix} I & -BR^{-1} \\ & I \end{bmatrix} \begin{bmatrix} A & B \\ B^T & R \end{bmatrix} \begin{bmatrix} I & \\ -R^{-1}B^T & I \end{bmatrix} = \begin{bmatrix} A - BR^{-1}B^T & \\ & R \end{bmatrix}$$

LMIs

For continuous-time systems

$$\begin{bmatrix} PA + A^T P & PB \\ B^T P & -\gamma^2 I \end{bmatrix} + \begin{bmatrix} C^T \\ D^T \end{bmatrix} \begin{bmatrix} C & D \end{bmatrix} \prec 0$$

or

$$\begin{bmatrix} PA + A^T P & PB & C^T \\ B^T P & -\gamma & D^T \\ C & D & -\gamma \end{bmatrix} \prec 0$$

The smallest γ for which we can find such a $P = P^T \succ 0$ gives the upper bound of $\|G\|_\infty$.