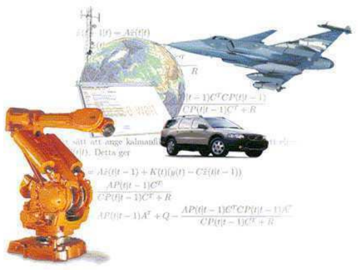
Robust Multivariable Control

Lecture 3



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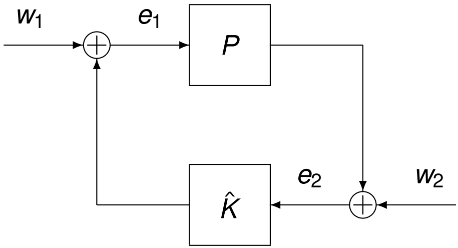
Today's topics

- Feedback
- Stability
- Stabilizability and Detectability
- Specifications
- Input-Output Duality
- Reformulation H_∞ norm

Feedback



$\hat{K} = -K$

(i) Negative feedback: $(I + PK)^{-1}$

(ii) Positive feedback: $(I - P\hat{K})^{-1}$

Well posed

$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \to \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}, \text{ exists and is proper}$$

$\Leftrightarrow$

$$I - \hat{K}(\infty)P(\infty) \quad \text{invertible}$$

or

$$I - P(\infty)\hat{K}(\infty) \quad \text{invertible}$$

or

$$\begin{bmatrix} I & -\hat{K}(\infty) \\ -P(\infty) & I \end{bmatrix} \quad \text{invertible}$$

Note: well-posed is not the same as stability.

Well posed

Let

$$P = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \quad \hat{K} = \left[\begin{array}{c|c} \hat{A} & \hat{B} \\ \hline \hat{C} & \hat{D} \end{array}\right]$$

Well posed

$$I - D\hat{D} \quad \text{invertible}$$

or

$$I - \hat{D}D \quad \text{invertible}$$

or

$$\begin{bmatrix} I & -\hat{D} \\ -D & I \end{bmatrix} \quad \text{invertible}$$

Stability

$$P:\begin{cases}\dot{x} &= Ax + Be_1\\ e_2 &= Cx + De_1 + w_2\end{cases}$$

$$\hat{K}:\begin{cases}\dot{\hat{x}} &= \hat{A}\hat{x} + \hat{B}e_2\\ e_1 &= \hat{C}\hat{x} + \hat{D}e_2 + w_1\end{cases}$$

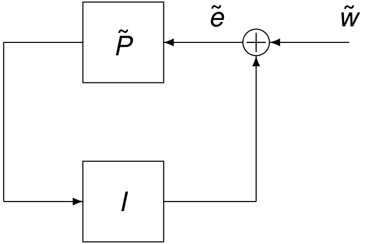
Thus,

$$\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} 0 & \hat{C} \\ C & 0 \end{bmatrix}\begin{bmatrix} x \\ \hat{x} \end{bmatrix} + \begin{bmatrix} 0 & \hat{D} \\ D & 0 \end{bmatrix}\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} + \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

$$\begin{bmatrix} I & -\hat{D} \\ -D & I \end{bmatrix}\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} 0 & \hat{C} \\ C & 0 \end{bmatrix}\begin{bmatrix} x \\ \hat{x} \end{bmatrix} + \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

$$\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \underbrace{\begin{bmatrix} I & -\hat{D} \\ -D & I \end{bmatrix}^{-1}}_{\text{well-posed}}\left(\begin{bmatrix} 0 & \hat{C} \\ C & 0 \end{bmatrix}\begin{bmatrix} x \\ \hat{x} \end{bmatrix} + \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}\right)$$

Special case $\hat{K} = I$



(*)

$$\tilde{P}: \begin{cases} \dot{\tilde{x}} &= \tilde{A}\tilde{x} + \tilde{B}\tilde{e} \\ \tilde{e} &= \tilde{C}\tilde{x} + \tilde{D}\tilde{e} + \tilde{w} \end{cases}$$

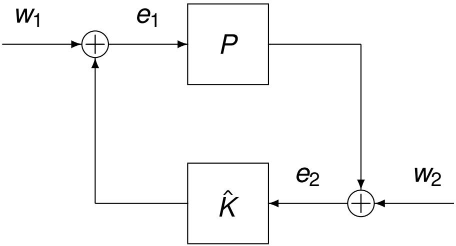
Special case $\hat{K} = I$

$$
\begin{aligned}
(I-\tilde{D})\tilde{e} &= \tilde{C}\tilde{x} + \tilde{w} \\
\tilde{e} &= (I-\tilde{D})^{-1}(\tilde{C}\tilde{x}+\tilde{w}) \\
\dot{\tilde{x}} &= \tilde{A}\tilde{x} + \tilde{B}(I-\tilde{D})^{-1}(\tilde{C}\tilde{x}+\tilde{w}) \\
\dot{\tilde{x}} &= (\tilde{A}+\tilde{B}(I-\tilde{D})^{-1}\tilde{C})\tilde{x} + \tilde{B}(I-\tilde{D})^{-1}\tilde{w}
\end{aligned}
$$

$$
\left[\begin{array}{c|c} \tilde{A} & \tilde{B} \\ \hline -\tilde{C} & I-\tilde{D} \end{array}\right]^{-1} = \left[\begin{array}{c|c} \tilde{A}+\tilde{B}(I-\tilde{D})^{-1}\tilde{C} & * \\ \hline * & * \end{array}\right]
$$

Thus, (*) is stable if $(I-\tilde{P})^{-1} \in RH_\infty$
(R = real and rational, H_∞ = stable and limited in RHP)

General case



(**)

Stabilizability and Detectability

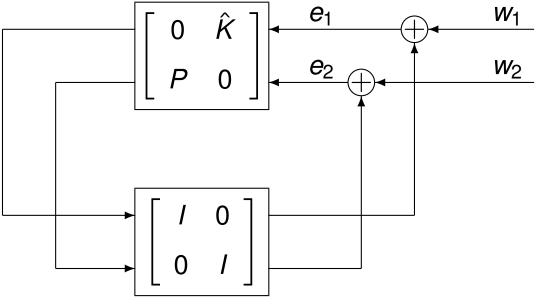
Requirements for stabilizing a feedback system.

Suppose that P and $\hat{K}$ are stabilizable and detectable.

Stabilizable: all *un*controllable modes are stable.

Detectable: all *un*observable modes are stable.

General case



General case

$$\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} + \begin{bmatrix} 0 & \hat{K} \\ P & 0 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$$

$$\begin{bmatrix} I & -\hat{K} \\ -P & I \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

$$\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} I & -\hat{K} \\ -P & I \end{bmatrix}^{-1} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

Consequently, (**) is stable if

$$\begin{bmatrix} I & -\hat{K} \\ -P & I \end{bmatrix}^{-1} \in RH_\infty$$

General case

It is not enough to only check that $(I - P\hat{K})^{-1}$ is stable.

For instance

$$P = \frac{s-1}{s+1}, \quad \hat{K} = -\frac{1}{s-1}$$

$$I - P\hat{K} = 1 + \frac{s-1}{s+1}\frac{1}{s-1} = \frac{2+s}{s+1}$$

Stability

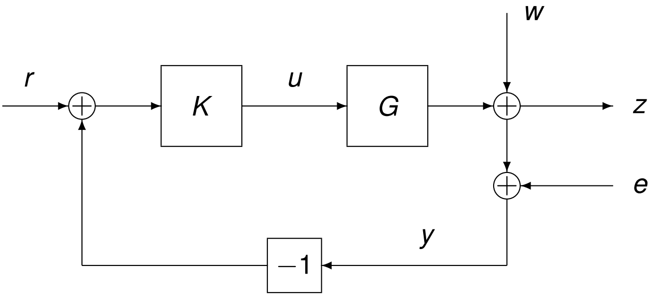
(i) Well-posed

(ii)

$$\begin{bmatrix} I & -\hat{K} \\ -P & I \end{bmatrix}^{-1} \in RH_\infty$$

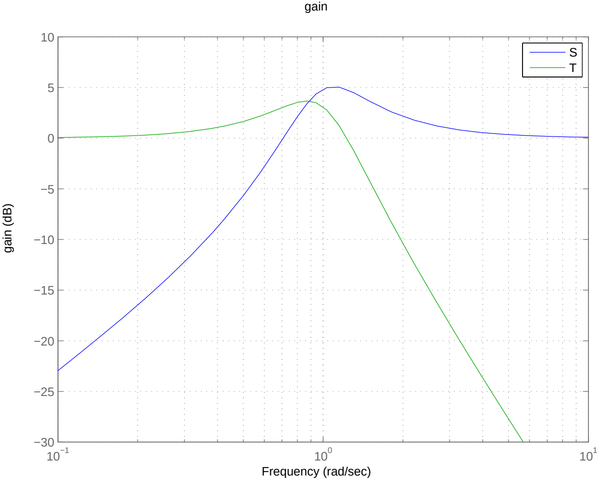
The *A*-matrix has all its eigenvalues in the LHP.

Performance specifications



$$z = Sw + T(r - e),$$
$$S = (I + GK)^{-1},$$
$$T = GK(I + GK)^{-1},$$
$$S + T = I$$

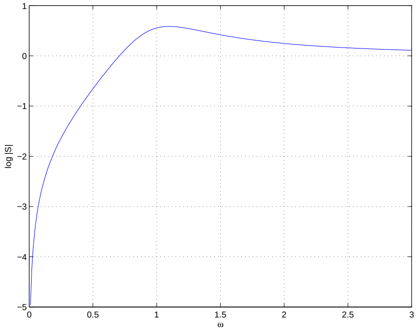
Performance specifications



Bode's sensitivity integral

If L is stable and has two more poles than zeros (note: linear frequency scale):

$$\int_0^\infty \log|S(j\omega)|d\omega = 0$$



Bode's integral relation

General:

$$\int_0^\infty \log|S(j\omega)|d\omega = \pi \sum_{\operatorname{Re} p_i \geq 0} \operatorname{Re} p_i$$

MIMO

$$\int_0^\infty \log \bar{\sigma}\left(S(j\omega)\right) d\omega \geq \begin{cases} \pi \max\limits_{\operatorname{Re} p_i \geq 0} \operatorname{Re} p_i \\ \pi \sum\limits_{\operatorname{Re} p_i \geq 0} \operatorname{Re} p_i \end{cases}$$

What about LQR? (60 deg, 6 dB margins)

Loop gain

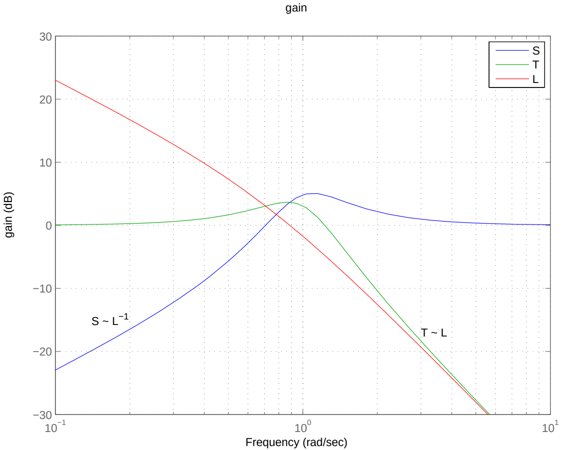
Loop gain: $L = GK$.

$S = (I + L)^{-1}$

$T = L(I + L)^{-1}$

Specify S and T $\Rightarrow$ specify L.

Loop gain


gain
30
20
10
0
−10
−20
−30
10^{-1}
10^{0}
10^{1}
gain (dB)
Frequency (rad/sec)
S
T
L
$S \sim L^{-1}$
$T \sim L$

Minimum phase

Minimum phase: relation between slope and phase

Slope $\|L\|$	phase
-1	-90 deg
-2	-180 deg

Non-minimum phase:

$$L = \frac{s-1}{s+2} = \underbrace{\frac{s-1}{s+1}}_{\text{all pass}} \times \underbrace{\frac{s+1}{s+2}}_{\text{min phase}}$$

$$\arg \frac{j\omega - 1}{j\omega + 1} = 2 \arctan \omega$$

Zeros in the RHP give worse phase margins at high frequencies
This leads to reduced bandwidth.

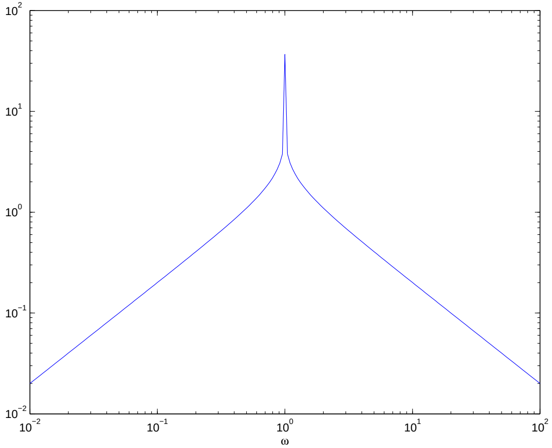
Non-minimum phase

Non-minimum phase can be interpreted as a time delay (all-pass), which results in reduced bandwidth.

$$\arg L(j\omega_0) = \frac{1}{\pi}\int \frac{d\log|L|}{d\nu}\log\coth\frac{|\nu|}{2}d\nu + \arg\prod_{\operatorname{Re} z_i \geq 0}\frac{j\omega_0 + z_i}{j\omega_0 - z_i}$$

where $\nu = \log(\omega/\omega_0)$.

$\log \coth \frac{|\log \omega|}{2}$



Multivariable

- $|S|$ is replaced by $\bar{\sigma}(S)$;
- $|T|$ is replaced by $\bar{\sigma}(T)$;
- $|L|$ is replaced by $\bar{\sigma}(L)$ and $\underline{\sigma}(L)$.

L can be either $L_o = GK$ or $L_i = KG$ depending on where the loop is opened.

The same applies to S and T.

Multivariable

Rewrite the sensitivity function, S,

$$\bar{\sigma}(S) = \bar{\sigma}\left((I+GK)^{-1}\right)$$
$$= \left(\underline{\sigma}(I+GK)\right)^{-1} \leq \frac{1}{\underline{\sigma}(GK)-1}$$

We can show that

$$\frac{1}{\underline{\sigma}(L)+1} \leq \bar{\sigma}(S) \leq \frac{1}{\underline{\sigma}(L)-1}$$

Multivariable

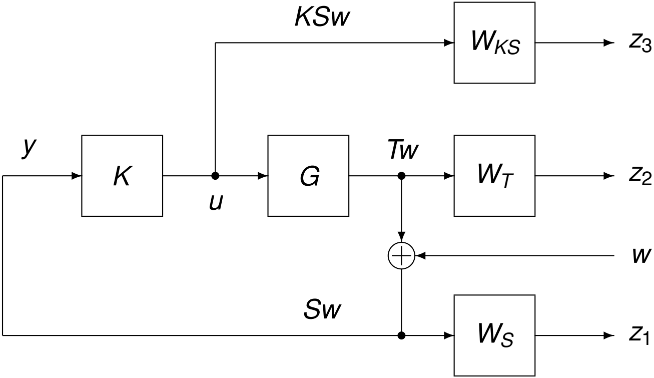
In the same way for T:

$$\begin{aligned}\bar{\sigma}(T) &= \bar{\sigma}\left(GK(I+GK)^{-1}\right) = \bar{\sigma}\left(L(I+L)^{-1}\right) \\ &= \bar{\sigma}\left(\left(L^{-1}+I\right)^{-1}\right) = \frac{1}{\underline{\sigma}\left(\left(L^{-1}+I\right)\right)} \\ &\leq \frac{1}{\underline{\sigma}\left(L^{-1}\right)-1} = \frac{\bar{\sigma}(L)}{1-\bar{\sigma}(L)}\end{aligned}$$

Thus,

$$\frac{\bar{\sigma}(L)}{1+\bar{\sigma}(L)} \leq \bar{\sigma}(T) \leq \frac{\bar{\sigma}(L)}{1-\bar{\sigma}(L)}$$

Performance specifications



Summary

$$\bar{\sigma}(S) \leq \frac{1}{\underline{\sigma}(L) - 1} \quad \text{try to make } \underline{\sigma}(L) \text{ large}$$

$$\bar{\sigma}(T) \leq \frac{\bar{\sigma}(L)}{1 - \bar{\sigma}(L)} \quad \text{try to make } \bar{\sigma}(L) \text{ small}$$

Consequently, squeeze $\bar{\sigma}(L)$ och $\underline{\sigma}(L)$ together at least for frequencies close to the cross-over frequency.

Try to have cross-over frequencies close to each other.

Input-output duality

We will often make use of the input-output duality in this course
For a state-space formulation

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \sim \begin{bmatrix} A^T & C^T \\ B^T & D^T \end{bmatrix}$$

Input-output duality

In discrete time

$$\begin{bmatrix} y(1) \\ y(2) \\ y(3) \\ \vdots \\ y(n) \end{bmatrix} = \begin{bmatrix} D & & & & \\ CB & D & & & \\ CAB & CB & D & & \\ \vdots & \ddots & \ddots & \ddots & \\ CA^{n-1}B & \dots & CAB & CB & D \end{bmatrix} \begin{bmatrix} u(1) \\ u(2) \\ u(3) \\ \vdots \\ u(n) \end{bmatrix}$$

where we have stacked the inputs and outputs into long vectors.

Swapping inputs with outputs results in a transposed matrix, or,

$$\begin{bmatrix} \upsilon(1) \\ \upsilon(2) \\ \upsilon(3) \\ \vdots \\ \upsilon(n) \end{bmatrix} = \begin{bmatrix} D^T & B^TC^T & B^TA^TC^T & \cdots & B^T(A^T)^{n-1}C^T \\ & D^T & B^TC^T & \ddots & \vdots \\ & & D^T & \ddots & B^TA^TC^T \\ & & & \ddots & B^TC^T \\ & & & & D^T \end{bmatrix} \begin{bmatrix} \zeta(1) \\ \zeta(2) \\ \zeta(3) \\ \vdots \\ \zeta(n) \end{bmatrix}$$

Input-output duality

Reverse time

$$\begin{bmatrix} \upsilon(n) \\ \upsilon(n-1) \\ \upsilon(n-2) \\ \vdots \\ \upsilon(1) \end{bmatrix} = \begin{bmatrix} D^T & & & & \\ B^T C^T & D^T & & & \\ B^T A^T C^T & B^T C^T & D^T & & \\ \vdots & \ddots & \ddots & \ddots & \\ B^T (A^T)^{n-1} C^T & \cdots & B^T A^T C^T & B^T C^T & D^T \end{bmatrix} \begin{bmatrix} \zeta(n) \\ \zeta(n-1) \\ y(n-2) \\ \vdots \\ \zeta(1) \end{bmatrix}$$

Which corresponds to the dual system

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \sim \begin{bmatrix} A^T & C^T \\ B^T & D^T \end{bmatrix}$$

Reformulating the H_∞ norm

The Hamiltonian,

$$H = \begin{bmatrix} A & 0 \\ -C^T C & -A^T \end{bmatrix} - \begin{bmatrix} B \\ -C^T D \end{bmatrix} \underbrace{\left(\gamma^2 I - D^T D\right)}_{R \succ 0}^{-1} \begin{bmatrix} -D^T C & -B^T \end{bmatrix}$$

$$= \begin{bmatrix} A + BR^{-1} D^T C & BR^{-1} B^T \\ -C^T C - C^T D R^{-1} D^T C & -A^T - C^T D R^{-1} B^T \end{bmatrix}$$

If H has no eigenvalues on the imaginary axis then $\gamma > \|G\|_\infty$.

Reformulating the H_∞ norm

Rewrite the eigenvalue problem, $Hx = \lambda x$ as $(H - \lambda I)x = 0$

$$H - \lambda I$$
$$= \begin{bmatrix} A - \lambda I & 0 \\ -C^T C & -A^T - \lambda I \end{bmatrix} - \begin{bmatrix} B \\ -C^T D \end{bmatrix} \underbrace{\left(\gamma^2 - D^T D\right)}_{R \succ 0}^{-1} \begin{bmatrix} -D^T C & -B^T \end{bmatrix}$$

Apply Schur complement

$$\sim \begin{bmatrix} A - \lambda I & 0 & B \\ -C^T C & -A^T - \lambda I & -C^T D \\ D^T C & B^T & -(\gamma^2 - D^T D) \end{bmatrix}$$

We know that $(\gamma^2 - D^T D)$ is non-singular if $\gamma > \|G\|_\infty$.

Reformulating the H_∞ norm

$$\sim \begin{bmatrix} A-\lambda I & 0 & B \\ -C^T C & -A^T-\lambda I & -C^T D \\ D^T C & B^T & D^T D-\gamma I \end{bmatrix}$$

and, another Schur complement:

$$\sim \begin{bmatrix} A-\lambda I & 0 & B & \\ & -A^T-\lambda I & & -C^T \\ & B^T & -\gamma^2 & D^T \\ C & & D & -I \end{bmatrix}$$

Reshuffle:

$$\sim \begin{bmatrix} & & A-\lambda I & B \\ & -I & C & D \\ -A^T-\lambda I & -C^T & & \\ B^T & D^T & & -\gamma^2 \end{bmatrix}$$

Reformulating the H_∞ norm

$$\sim \begin{bmatrix} & & A-\lambda I & B \\ & -I & C & D \\ -A^T-\lambda I & -C^T & & \\ B^T & D^T & & -\gamma^2 \end{bmatrix}$$

and rescale

$$\sim \left[\begin{array}{cc|cc} & & A & B \\ & -\gamma & C & D \\ \hline A^T & C^T & & \\ B^T & D^T & & -\gamma \end{array}\right] - \lambda \left[\begin{array}{c|c} & I \\ \hline -I & \end{array}\right]$$

This is an generalized eigenvalue problem, $(\mathscr{A} - \lambda\mathscr{E})x = 0$, with a structure similar to the original system.