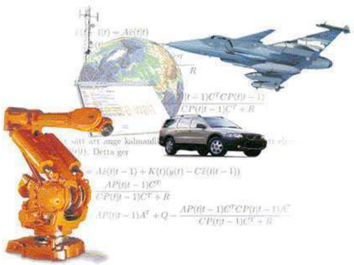
Robust Multivariable Control

Lecture 7



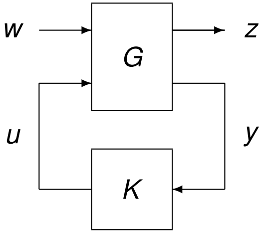
Anders Helmersson
anders.helmersson@liu.se

ISY/Reglerteknik
Linköpings universitet

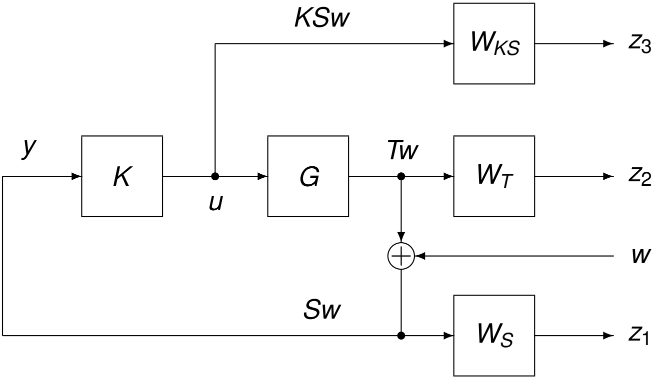
Today's topics

- Requirements
- Coprime factorization
- Uncertainties
- Synthesis
- Methodology
- Model reduction
- ν-gap

H_∞ problem



H_∞ in closed loop



Requirements

Requirements in terms of W_S, W_T och W_{KS}:

- $\bar{\sigma}(S(j\omega) < 1/\underline{\sigma}(W_S(j\omega))$
- $\bar{\sigma}(T(j\omega)) < 1/\underline{\sigma}(W_T(j\omega))$
- $\bar{\sigma}(KS(j\omega)) < 1/\underline{\sigma}(W_{KS}(j\omega))$

which can be rewritten as

- $\|W_S S\|_\infty < 1$
- $\|W_T T\|_\infty < 1$
- $\|W_{KS} KS\|_\infty < 1$

A certain conservatism (up to a factor $\sqrt{3}$, since $\|[1\ 1\ 1]\| = \sqrt{3}$).

Loop gain

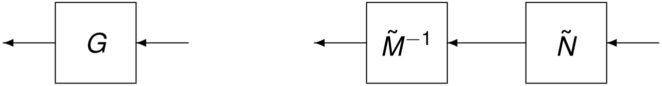
We can also define requirements on the loop gain, $L_o = GK$ or $L_i = KG$:

$$L_L(\omega) \leq \sigma_i(L(j\omega)) \leq U_L(\omega)$$

Compare with lead-lag design in the SISO case.

Coprime factorization

We will use the left coprime factorization.



$$G = \tilde{M}^{-1}\tilde{N}$$

Normalized if $\tilde{M}(s)\tilde{M}^T(-s) + \tilde{N}(s)\tilde{N}^T(-s) = I$, which implies

$$\left\| \begin{bmatrix} \tilde{M} & \tilde{N} \end{bmatrix} \right\|_\infty = 1.$$

Use for instance `ncfmr` in Matlab.

Anders Helmersson anders.helmersson@liu.se

REGLERTEKNIK
KOMMUNIKATIONSSYSTEM
LINKÖPINGS TEKNISKA HÖGSKOLA

Factorization

Let $G = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$

Choose L so that $A + LC$ becomes stable.

Then

$$\begin{bmatrix} \tilde{M} & \tilde{N} \end{bmatrix} = \left[\begin{array}{c|cc} A+LC & L & B+LD \\ \hline C & I & D \end{array}\right]$$

is a comprime factorization of G.

Normalized factorization

Assume that $D = 0$ and let $L = -YC^T$ where Y is a stabilizing solution to the Riccati equation:

$$AY + YA^T - YC^TCY + BB^T = 0.$$

Choose

$$\begin{bmatrix} \tilde{M} & \tilde{N} \end{bmatrix} = \left[\begin{array}{c|cc} A+LC & L & B \\ \hline C & I & 0 \end{array}\right]$$

Then

$$\begin{bmatrix} \tilde{M} & \tilde{N} \end{bmatrix}\begin{bmatrix} \tilde{M} & \tilde{N} \end{bmatrix}^{\sim} = \left[\begin{array}{cc|c} -(A-YC^TC)^T & 0 & -C^T \\ BB^T + YC^TCY & A - YC^TC & -YC^T \\ \hline -CY & C & I \end{array}\right]$$

where $M^{\sim}(s) = M(-s)^T$.

Normalization

Use

$$T = \left[\begin{array}{cc|c} I & 0 & \\ Y & I & \\ \hline & & I \end{array}\right] = \left[\begin{array}{cc|c} I & 0 & \\ -Y & I & \\ \hline & & I \end{array}\right]^{-1}$$

as a similarity transformation.

$$\begin{aligned}
\left[\begin{array}{cc} \tilde{M} & \tilde{N} \end{array}\right] \left[\begin{array}{c} \tilde{M}^{\sim} \\ \tilde{N}^{\sim} \end{array}\right] &\sim T^{-1} \left[\begin{array}{cc|c} -(A-YC^TC)^T & 0 & -C^T \\ BB^T + YC^TCY & A - YC^TC & -YC^T \\ \hline -CY & C & I \end{array}\right] T \\
&= \left[\begin{array}{cc|c} -(A-YC^TC)^T & 0 & -C^T \\ YA^T + BB^T & A - YC^TC & 0 \\ \hline -CY & C & I \end{array}\right] T \\
&= \left[\begin{array}{cc|c} -(A-YC^TC)^T & 0 & -C^T \\ AY + YA^T - YC^TCY + BB^T & A - YC^TC & 0 \\ \hline 0 & C & I \end{array}\right] \sim I.
\end{aligned}$$

Robustness

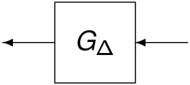
A robust controller should cope with uncertainties in the process

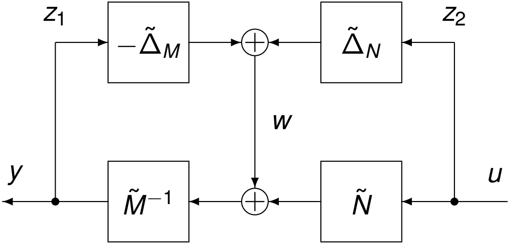
$$G_\Delta = \left(\tilde{M} + \tilde{\Delta}_M\right)^{-1} \left(\tilde{N} + \tilde{\Delta}_N\right)$$

where we assume that $\left\| \begin{bmatrix} \tilde{\Delta}_M & \tilde{\Delta}_N \end{bmatrix} \right\|_\infty \leq \varepsilon$.

Note that both the denominator and numerator may change. We have uncertainties in both *poles* och *zeros*.

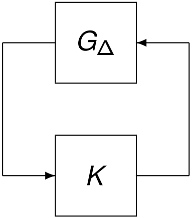
Description of uncertainties

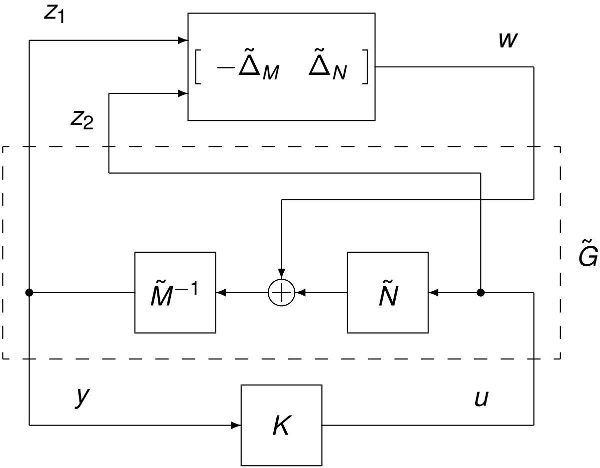




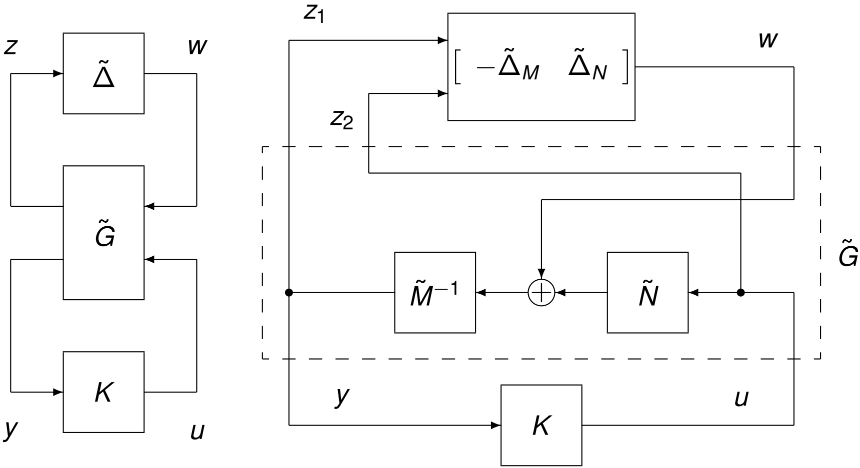
$$G_\Delta = \left(\tilde{M} + \tilde{\Delta}_M\right)^{-1} \left(\tilde{N} + \tilde{\Delta}_N\right)$$

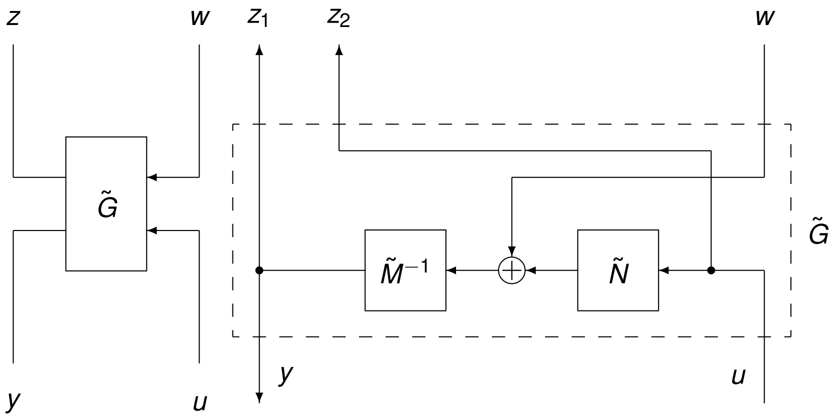
Stabilize G_Δ with a controller





$\tilde{G}$

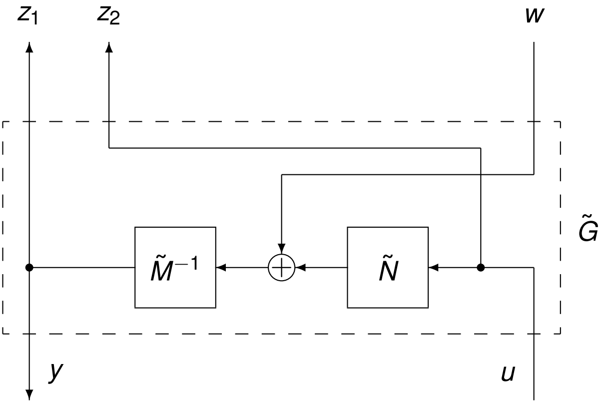

z
w
$\tilde{\Delta}$
$\tilde{G}$
K
y
u
z_1
$[\; -\tilde{\Delta}_M \quad \tilde{\Delta}_N \;]$
w
z_2
$\tilde{M}^{-1}$
$\tilde{N}$
$\tilde{G}$
y
K
u



z
w
y
u
G̃
z1
z2
w
M̃−1
Ñ
y
u
G̃

$\tilde{G}$

$$\tilde{G} = \left[\begin{array}{c|c} \tilde{M}^{-1} & G \\ 0 & I \\ \hline \tilde{M}^{-1} & G \end{array}\right]$$



Synthesis

$$\tilde{G} = \left[\begin{array}{c|c} \tilde{M}^{-1} & G \\ 0 & I \\ \hline \tilde{M}^{-1} & G \end{array}\right] = \left[\begin{array}{c|c|c} A & -L & B \\ \hline C & I & 0 \\ 0 & 0 & I \\ \hline C & I & 0 \end{array}\right] \qquad \text{with } L = -YC^T$$

where $Y \succeq 0$ is a stabilizing solution to

$$AY + YA^T - YC^TCY + BB^T = 0$$

We can show that $\gamma_{\min} = \sqrt{1 + \lambda_{\max}(XY)}$, where $X \succeq 0$ is a stabilizing solution to

$$XA + A^TX - XBB^TX + C^TC = 0$$

$\tilde{M}^{-1}$

$$\tilde{M}^{-1} = \left[\begin{array}{c|c} A+LC & L \\ \hline C & I \end{array}\right]^{-1} = \left[\begin{array}{c|c} A+LC-LC & -L \\ \hline C & I \end{array}\right] = \left[\begin{array}{c|c} A & -L \\ \hline C & I \end{array}\right]$$

$D_{11} \neq 0$

$$\tilde{G} = \left[\begin{array}{c|c|c} A & -L & B \\ \hline C & I & 0 \\ 0 & 0 & I \\ \hline C & I & 0 \end{array}\right] = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right]$$

Here $D_{11} \neq 0$. How can we solve this?

Idea: if $\|D_{11}\| < 1$ then

$$N = \begin{bmatrix} -D_{11} & (I - D_{11}D_{11}^T)^{1/2} \\ (I - D_{11}^T D_{11})^{1/2} & D_{11}^T \end{bmatrix}$$

is a unitary matrix: $N^T N = I$.

$D_{11} \neq 0$

Let

$$\begin{bmatrix} r \\ w \end{bmatrix} = N \begin{bmatrix} v \\ z \end{bmatrix}$$

$$\begin{bmatrix} r \\ w \end{bmatrix}^T \begin{bmatrix} r \\ w \end{bmatrix} = \begin{bmatrix} v \\ z \end{bmatrix}^T \underbrace{N^T N}_{=I} \begin{bmatrix} v \\ z \end{bmatrix}$$

Thus,

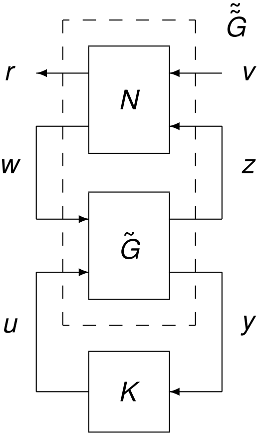
$$r^T r + w^T w = v^T v + z^T z$$

and

$$\|r\|_2^2 - \|v\|_2^2 = \|z\|_2^2 - \|w\|_2^2$$

That is to say, if the H_∞ gain from w to z is less than one, then also the gain from v to r is less than one.

$D_{11} \neq 0$



Choose

$$N = \begin{bmatrix} -D_{11} & (I - D_{11} D_{11}^T)^{1/2} \\ (I - D_{11}^T D_{11})^{1/2} & D_{11}^T \end{bmatrix}$$

(Then the original D_{11} disappears in $\tilde{\tilde{G}}$.

Note that this assumes that we have applied a scaling of the original system so that γ becomes one.

Synthesis

Scale the system:

$$\tilde{G} = \left[\begin{array}{c|c|c} A & -L & B \\ \hline \gamma^{-1}C & \gamma^{-1}I & 0 \\ 0 & 0 & \gamma^{-1}I \\ \hline C & I & 0 \end{array}\right]$$

Apply N:

$$\tilde{\tilde{G}} = \left[\begin{array}{c|c|c} A - \frac{1}{\gamma^2-1}LC & -\frac{\gamma}{\sqrt{\gamma^2-1}}L & B \\ \hline \frac{1}{\sqrt{\gamma^2-1}}C & 0 & 0 \\ 0 & 0 & \gamma^{-1}I \\ \hline \frac{\gamma^2}{\gamma^2-1}C & \frac{\gamma}{\sqrt{\gamma^2-1}}I & 0 \end{array}\right]$$

Synthesis

Then, scale u and y such that $D_{12}^T D_{12} = I$ and $D_{21} D_{21}^T = I$:

$$\tilde{\tilde{G}} = \left[\begin{array}{c|c|c} A - \frac{1}{\gamma^2-1} LC & -\frac{\gamma}{\sqrt{\gamma^2-1}} L & \gamma B \\ \hline \frac{1}{\sqrt{\gamma^2-1}} C & 0 & 0 \\ 0 & 0 & I \\ \hline \frac{\gamma}{\sqrt{\gamma^2-1}} C & I & 0 \end{array}\right]$$

The Riccati equation (DF) is defined by (with $L = -YC^T$ and $\tilde{\tilde{\gamma}} = 1$):

$$H = \begin{bmatrix} A + \frac{1}{\gamma^2-1} YC^T C & \frac{\gamma^2}{\gamma^2-1} YC^T CY - \gamma^2 BB^T \\ -\frac{1}{\gamma^2-1} C^T C & -(A + \frac{1}{\gamma^2-1} YC^T C)^T \end{bmatrix}$$

Synthesis

Transform H:

$$\begin{bmatrix} I & Y \\ 0 & (\gamma^2-1)I \end{bmatrix} \begin{bmatrix} A+\frac{1}{\gamma^2-1}YC^TC & \frac{\gamma^2}{\gamma^2-1}YC^TCY-\gamma^2BB^T \\ -\frac{1}{\gamma^2-1}C^TC & -(\gamma^2-1)A^T-C^TCY \end{bmatrix} \begin{bmatrix} I & -\frac{1}{\gamma^2-1}Y \\ 0 & \frac{1}{\gamma^2-1}I \end{bmatrix}$$

$$= \begin{bmatrix} A & \frac{\gamma^2}{\gamma^2-1}YC^TCY-\gamma^2BB^T-YA^T-\frac{1}{\gamma^2-1}YC^TCY \\ -C^TC & -A^T-\frac{1}{\gamma^2-1}C^TCY \end{bmatrix} \begin{bmatrix} I & -\frac{1}{\gamma^2-1}Y \\ 0 & \frac{1}{\gamma^2-1}I \end{bmatrix}$$

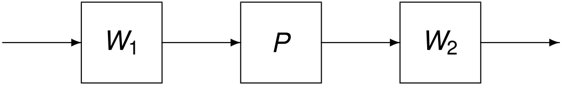
$$= \begin{bmatrix} A & \frac{1}{\gamma^2-1}(-AY-YA^T+YC^TCY)-\frac{\gamma^2}{\gamma^2-1}BB^T \\ -C^TC & -A^T \end{bmatrix}$$

$$= \begin{bmatrix} A & BB^T \\ -C^TC & -A^T \end{bmatrix} \quad \text{corresponding to} \quad XA+A^TX-XBB^TX+C^TC=0$$

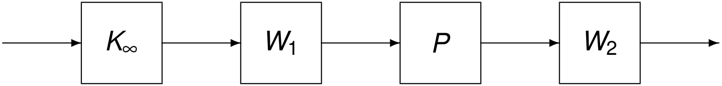
In addition, we have a condition that $I-\frac{1}{\gamma^2-1}YX$ must be invertible, which yields $\gamma_{\min}^2-1=\lambda_{\max}(YX)$, or $\gamma_{\min}=\sqrt{1+\lambda_{\max}(XY)}$.

Loop shaping – Methodology

1) Scale the system P with W_1 och W_2 so that the desired gain is obtained for the open-loop system.



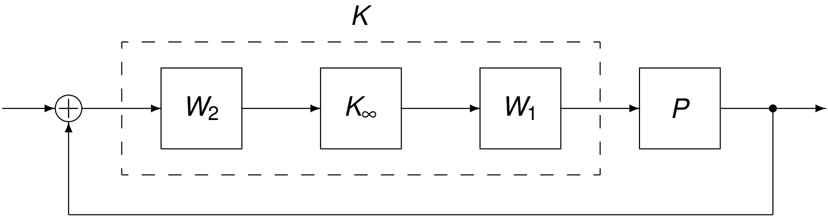
2) Compute $\varepsilon_{\max} = \frac{1}{\gamma_{\min}}$ and K_∞:



If $\varepsilon_{\max}$ is small $< 0.2 - 0.3$ then go back to 1) and adjust the requirements on W_1 and W_2.

Loop shaping – Methodology

3) Compute the final controller: $K = W_1 K_\infty W_2$:



Typically, integrators are added into W_1 (or W_2).

Try to squeeze the gains (σ_i) together close to the cross-over frequency.

Use for instance `ncfsyn` in Matlab.

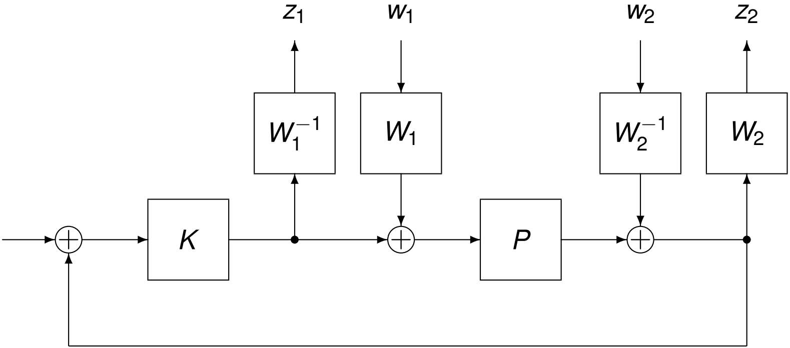
Loop shaping – Methodology

Observe

The phase is taken care of automatically in the synthesis step.

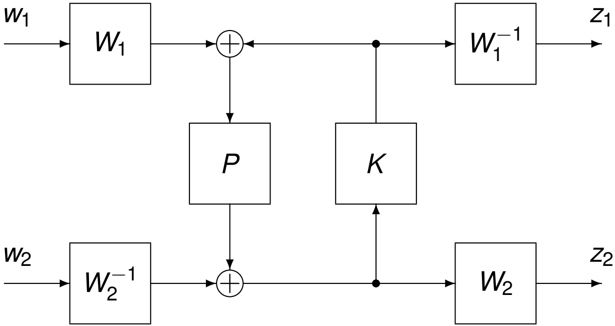
In the design step (choice of W_1 and W_2) we only consider the gain. W_i adjusts the slope.

Alternative formulation



Anders Helmersson anders.helmersson@liu.se

Alternative formulation



Margins

$\varepsilon = 0.3$ corresponds to about 17 degrees phase margin and 2.7 dB gain margin on the input *and* output.

For the SISO case, 35 degrees and 5.4 dB phase and gain margin.

Loop shaping

Pros:

- Simple design
- Easy to get started

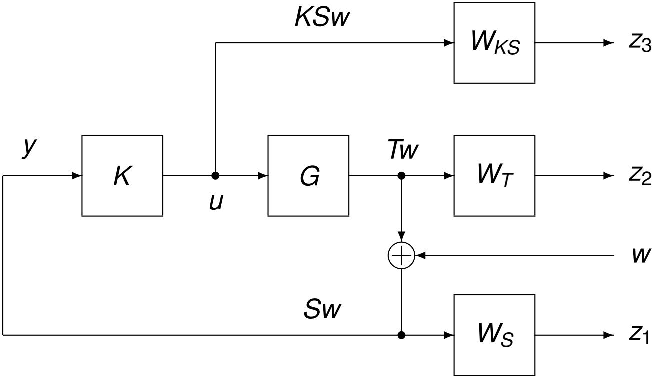
Con:

- Difficult to express different requirements for the different channels

Tip: try to reduce the condition number of W_1 and W_2 for multivariable systems.

Model reduction of controllers

Motivation:



The states of the plant and weights will show up in the controller.

Model reduction

Keep the number of states in the controller, K, as low as possible.

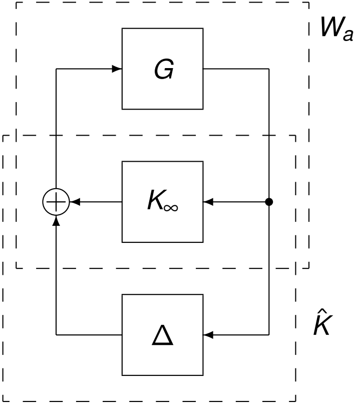
Easier to implement, easier to understand (perhaps) and less complicated.

How to guarantee stability and performance for a reducered order controller, $\hat{K}$.

$\|K_\infty - \hat{K}\|_\infty$ must be is sufficiently small.

Stability

Let $\Delta = \hat{K} - K_\infty$:



Stability

Stable if Δ stable and if $\|W_a \Delta\|_\infty < 1$ or if $\|\Delta W_a\|_\infty < 1$ where

$$W_a = (I - GK_\infty)^{-1} G$$

Weighted model reduction [see ZDG, 7.2].

Stability of closed-loop coprime-factorized system

Let $G = NM^{-1} = \tilde{M}^{-1}\tilde{N}$ och $K = UV^{-1} = \tilde{V}^{-1}\tilde{U}$. The the following statements are equivalent:

(i) G closed with K is stable.

(ii) $\begin{bmatrix} M & U \\ N & V \end{bmatrix}$ is invertible in $\mathscr{R}\mathscr{H}_\infty$.

(iii) $\begin{bmatrix} \tilde{V} & -\tilde{U} \\ -\tilde{N} & \tilde{M} \end{bmatrix}$ invertible in $\mathscr{R}\mathscr{H}_\infty$.

(iv) $\tilde{V}N - \tilde{U}M$ invertible in $\mathscr{R}\mathscr{H}_\infty$.

(v) $\tilde{M}V - \tilde{N}U$ invertible in $\mathscr{R}\mathscr{H}_\infty$.

Stability of closed-loop coprime-factorized system

Proof (sketch):

$$\begin{bmatrix} I & K \\ G & I \end{bmatrix} = \begin{bmatrix} I & UV^{-1} \\ NM^{-1} & I \end{bmatrix} = \begin{bmatrix} M & U \\ N & V \end{bmatrix} \begin{bmatrix} M^{-1} & 0 \\ 0 & V^{-1} \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} I & K \\ G & I \end{bmatrix}^{-1}}_{\text{(i)}\Leftrightarrow(.)\in\mathcal{RH}_\infty} = \begin{bmatrix} M & 0 \\ 0 & V \end{bmatrix} \underbrace{\begin{bmatrix} M & U \\ N & V \end{bmatrix}^{-1}}_{\text{(ii)}\Leftrightarrow(.)\in\mathcal{RH}_\infty}$$

Further, using the same arguments,

$$\underbrace{\begin{bmatrix} \tilde{V} & -\tilde{U} \\ -\tilde{N} & \tilde{M} \end{bmatrix}}_{\text{(iii)}\Leftrightarrow(.)^{-1}\in\mathcal{RH}_\infty} \underbrace{\begin{bmatrix} M & U \\ N & V \end{bmatrix}}_{\text{(ii)}\Leftrightarrow(.)^{-1}\in\mathcal{RH}_\infty} = \underbrace{\begin{bmatrix} \tilde{V}M - \tilde{U}N & 0 \\ 0 & \tilde{M}V - \tilde{N}U \end{bmatrix}}_{\text{(iv) och (v)}\Leftrightarrow(.)^{-1}\in\mathcal{RH}_\infty}$$

Model reduction and coprime factorization

Let $G = \tilde{M}^{-1}\tilde{N}$, $K = UV^{-1}$ and $\hat{K} = \hat{U}\hat{V}^{-1}$.

G closed with $\hat{K}$ is stable if

$$\tilde{M}\hat{V} - \tilde{N}\hat{U} = \underbrace{\left(\tilde{M}V - \tilde{N}U\right)}_{(.)^{-1} \in \mathscr{RH}_\infty} \left(I - \begin{bmatrix} -\tilde{N}_n & \tilde{M}_n \end{bmatrix} \begin{bmatrix} U - \hat{U} \\ V - \hat{V} \end{bmatrix} \right) \in \mathscr{RH}_\infty$$

where $\begin{bmatrix} \tilde{N}_n & \tilde{M}_n \end{bmatrix} := \left(\tilde{M}V - \tilde{N}U\right)^{-1} \begin{bmatrix} \tilde{N} & \tilde{M} \end{bmatrix}$.

Thus

$$\left\| \begin{bmatrix} -\tilde{N}_n & \tilde{M}_n \end{bmatrix} \left(\begin{bmatrix} U \\ V \end{bmatrix} - \begin{bmatrix} \hat{U} \\ \hat{V} \end{bmatrix} \right) \right\|_\infty < 1$$

guarantees stability.

Weighted model reduction

General problem: $\|W_2^{-1}(K-\hat{K})W_1^{-1}\|_\infty < 1$.

Frequency weighted model reduction

$$\min_{\hat{G}} \|W_o(G-\hat{G})W_i\|_\infty$$

Idea: extend the system G with W_o and W_i:

$$W_oGW_i = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

Solve the Lyapunov equations: (i) P belongs to W_o and G (ii) Q belongs to W_i and G

$$TPT^T = T^{-T}QT^{-1} = \begin{bmatrix} \Sigma_1 & \\ & \Sigma_2 \end{bmatrix}$$

Some thoughts about model reduction

H_∞ design gives often a controller with many states, sometimes 50 to 100 states (weights + G).

Unrealistic to implement.

Step of reduction:

- Use balancing and truncation to get rid of states that do not affect the system response.
- Remove fast poles (much faster than the bandwidth of the system). Replace fast dynamics by a constant term.
- Then use more advanced methods, such as weighted model reduction.

New methods: for instance `hinfstruct` in Matlab.

hinfstruct

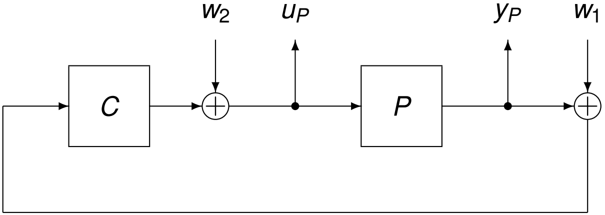
Example DLR1 in Compleib, $n = 10$, with $\gamma^* = 0.0619$.

- `[g, dim] = complib ('DLR1');`
- `[k, cl, gam] = hinfsyn (g, dim(1), dim(2));`
- $\gamma = 0.0625$
- `blk = ltiblock.ss ('demo', n, dim(2), dim(1));`
- `opts = hinfstructOptions ('RandomStart', 10);`
- `[cl, gam, info] = hinfstruct (lft (g, blk), opts);`
- `k = ss (cl.Blocks.demo);`
- $n = 5, \gamma = 0.0619$
- $n = 4, \gamma = 0.0742$
- $n = 3, \gamma = 0.2055$

ν-gap

$$b_{P,C} = \left\| \begin{bmatrix} P \\ I \end{bmatrix} (I - CP)^{-1} \begin{bmatrix} -C & I \end{bmatrix} \right\|_\infty^{-1} (\leq 1 \text{ always})$$

If $[P, C]$ is unstable then $b_{P,C} = 0$.



$$\begin{bmatrix} y_P \\ u_P \end{bmatrix} = \begin{bmatrix} P \\ I \end{bmatrix} (I - CP)^{-1} \begin{bmatrix} -C & I \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

$b_{P,C}$ is a generic index of closed loop performance.

ν-gap

The ν-gap is a measure of distance between two systems, P_1 and P_2.

$$\delta_\nu(P_1,P_2) := \begin{cases} \|\tilde{G}_2 G_1\|_\infty & \text{if}\, \det(G_2^* G_1)(j\omega) \neq 0,\ \text{wno}\det(G_2^* G_1) = 0 \\ 1 & \text{else} \end{cases}$$

where

$$G_i = \begin{bmatrix} N_i \\ M_i \end{bmatrix} \qquad \tilde{G}_i = \begin{bmatrix} -\tilde{M}_i & \tilde{N} \end{bmatrix}$$

$$P_i = N_i M_i^{-1} = \tilde{M}_i^{-1} \tilde{N}_i$$

$$\arcsin b_{P_2,C} \geq \arcsin b_{P_1,C} - \arcsin \delta_\nu(P_1,P_2)$$

ν-gap

Theorem

(i) Given P_1, C and β then $[P_2, C]$ is stable for all P_2 such that $\delta_\nu(P_1, P_2) \leq \beta$ if and only if $b_{P_1,C} > \beta$.

(ii) Given P_1, P_2 and β then $[P_2, C]$ is stable for all C such that $b_{P_1,C} > \beta$ if and only if $\delta_\nu(P_1, P_2) \leq \beta$.