# Iterative Learning Control – From a Controllability Point of View

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## 1. INTRODUCTION

Iterative learning control (ILC) is a method to improve the control of processes that perform the same task repeatedly [Arimoto et al., 1984]. A good example of such a process is an industrial robot performing arc welding or laser cutting in a general production situation. The system used for ILC can be both an open loop system as well as a closed loop system. Usually, the ILC control signal $\mathbf{u}_k(t) \in \mathbb{R}^{n_u}$ is updated according to

$$\mathbf{u}_{k+1}(t) = \mathcal{F}\left(\{\mathbf{u}_k(i)\}_{i=0}^{N-1}, \{\mathbf{e}_k(i)\}_{i=0}^{N-1}\right), \ t = 0, \dots, N-1,$$

where $\mathbf{e}_k(t) = \mathbf{r}(t) - \mathbf{y}_k(t)$ is the control error, $\mathbf{r}(t) \in \mathbb{R}^{n_y}$ the reference signal, $\mathbf{y}_k(t) \in \mathbb{R}^{n_y}$ the measurement signal, $k$ the iteration index, $t$ the time index and $\mathcal{F}(\,\cdot\,)$ is an update function. The main task is to find an update function that is able to drive the error to zero as the number of iterations tends to infinity, i.e.,

$$\|\mathbf{e}_k(t)\| \to 0, \quad k \to \infty, \forall t. \tag{1}$$

For the convergence proof it is usually suitable to use a batch description of the system given by

$$\overline{\mathbf{y}}_k = \mathbf{S}_\mathbf{u}\overline{\mathbf{u}}_k + \mathbf{S}_\mathbf{r}\overline{\mathbf{r}}. \tag{2}$$

In Lee and Lee [1998, 2000], Lee et al. [2000] it is proven that (1) holds under the assumption that $\mathbf{S}_\mathbf{u}$ has full row rank. Moreover, in Amann et al. [1996] it is assumed that $\ker \mathbf{S}_\mathbf{u}^\mathsf{T} = \emptyset$ which is equivalent to $\mathbf{S}_\mathbf{u}$ having full row rank. An important implication from this assumption is that it is necessary to have at least as many control signals as measurement signals. Even if the number of measurement signals and control signals are the same it can not be guaranteed that the full rank requirement is fulfilled.

The assumption that $\mathbf{S}_\mathbf{u}$ has full row rank will be investigated here, based on a state space model in the iteration domain for which output controllability is considered. The result shows that the requirement of full row rank of $\mathbf{S}_\mathbf{u}$ is equivalent to the proposed state space model being output controllable. The aspects of controllability are then extended to target path controllability (TPC) [Engwerda, 1988] which is shown to be a more suitable requirement for ILC. TPC naturally leads to the concept of "lead-in", which is about extending the trajectory with a part in the beginning, where it is not important to have perfect trajectory tracking.

## 2. PROPOSED STATE SPACE MODEL

Given the linear time-invariant state space model

$$\mathbf{x}(t+1) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}_\mathbf{u}\mathbf{u}(t) + \mathbf{B}_\mathbf{r}\mathbf{r}(t), \tag{3a}$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t), \tag{3b}$$

then the following batch formulation of the system can be obtained

$$\overline{\mathbf{x}} = \mathbf{\Phi}\mathbf{x}(0) + \mathbf{S}_\mathbf{xu}\overline{\mathbf{u}} + \mathbf{S}_\mathbf{xr}\overline{\mathbf{r}}, \tag{4a}$$

$$\overline{\mathbf{y}} = \boldsymbol{\mathcal{C}}\overline{\mathbf{x}}. \tag{4b}$$

where $\overline{\mathbf{x}} = \left(\mathbf{x}(1)^\mathsf{T} \ \dots \ \mathbf{x}(N)^\mathsf{T}\right)^\mathsf{T}$ and similar for $\overline{\mathbf{u}}$, $\overline{\mathbf{r}}$, and $\overline{\mathbf{y}}$. At ILC iteration $k$ and $k+1$ it holds that

$$\overline{\mathbf{x}}_k = \mathbf{\Phi}\mathbf{x}(0) + \mathbf{S}_\mathbf{xu}\overline{\mathbf{u}}_k + \mathbf{S}_\mathbf{xr}\overline{\mathbf{r}}, \tag{5}$$

$$\overline{\mathbf{x}}_{k+1} = \mathbf{\Phi}\mathbf{x}(0) + \mathbf{S}_\mathbf{xu}\overline{\mathbf{u}}_{k+1} + \mathbf{S}_\mathbf{xr}\overline{\mathbf{r}}. \tag{6}$$

Subtracting (5) from (6) gives

$$\overline{\mathbf{x}}_{k+1} = \overline{\mathbf{x}}_k + \mathbf{S}_\mathbf{xu}(\overline{\mathbf{u}}_{k+1} - \overline{\mathbf{u}}_k) = \overline{\mathbf{x}}_k + \mathbf{S}_\mathbf{xu}\Delta_{\overline{\mathbf{u}}_k}, \tag{7}$$

where $\Delta_{\overline{\mathbf{u}}_k} \triangleq \overline{\mathbf{u}}_{k+1} - \overline{\mathbf{u}}_k$ is considered as a new control signal. The state space model in the iteration domain is therefore given by (7) and (4b).

## 3. CONTROLLABILITY

An important property for state space models are controllability, which considers the ability to control the system to a predefined state or output. The controllability matrix $\mathscr{C}$ for the system in the iteration domain is given by $\mathbf{S}_\mathbf{xu}$ repeated $N$ times. Theorem 1 states a condition for controllability.

*Theorem 1.* The system in the iteration domain is controllable if and only if

$$\operatorname{rank}\mathscr{C} = \operatorname{rank}\left(\mathbf{S}_\mathbf{xu} \ \cdots \ \mathbf{S}_\mathbf{xu}\right) = \operatorname{rank}\mathbf{S}_\mathbf{xu} = Nn_x.$$

It follows from Theorem 1 that a necessary condition for the system to be controllable is that $n_u \geq n_x$. Often, it is not of interest in ILC to control all the states but only the output. Therefore, it is more relevant to consider output controllability of the system. The requirement for output controllability is that the output controllability matrix $\mathscr{C}^o = \boldsymbol{\mathcal{C}}\mathscr{C}$ has full rank [Ogata, 2002]. A condition for output controllability is stated in Theorem 2.

*Theorem 2.* The system in the iteration domain is output controllable if and only if

$$\operatorname{rank}\mathscr{C}^o = \operatorname{rank}\left(\boldsymbol{\mathcal{C}}\mathbf{S}_\mathbf{xu} \ \cdots \ \boldsymbol{\mathcal{C}}\mathbf{S}_\mathbf{xu}\right) = \operatorname{rank}\boldsymbol{\mathcal{C}}\mathbf{S}_\mathbf{xu} = Nn_y.$$

It can now be shown that a necessary condition for the system to be output controllable is that $\operatorname{rank}\mathbf{B}_\mathbf{u} \geq n_y$. In Section 1 it was mentioned that in Lee and Lee [1998, 2000], Lee et al. [2000], Amann et al. [1996] $\mathbf{S}_\mathbf{u}$ was assumed to have full row rank. In this work, $\mathbf{S}_\mathbf{u} = \boldsymbol{\mathcal{C}}\mathbf{S}_\mathbf{xu}$, hence the property (1) holds if the state space model in the iteration domain is output controllable.

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## 4. INTERPRETATION OF CONTROLLABILITY

A single input system with state dimension $n_x$ can require $n_x$ time steps to be able to reach the desired state $\mathbf{x}_f$ or the desired output $\mathbf{y}_f$. It means that it can take up to $n_x$ time steps before the state space model (3) reaches the reference trajectory. This practically means that the first part of $\overline{\mathbf{x}}$, and the corresponding part of $\overline{\mathbf{y}}$, cannot be defined by an arbitrary reference.

Another reason for the model not being controllable is the construction of the vectors $\overline{\mathbf{x}}$ and $\overline{\mathbf{y}}$. It will be physically impossible to achieve any given $\overline{\mathbf{x}}_f$ or $\overline{\mathbf{y}}_f$. A simple example with $n_x = 2$, where the states are position and velocity, and the input is the acceleration, will be used to illustrate this.

Let $p(t)$ be the position, $v(t)$ the velocity, and let the state vector be $\mathbf{x}(t) = (p(t)\ v(t))^{\mathsf{T}}$, then the discrete-time model, using zero order hold sampling, becomes

$$\mathbf{x}(t+1) = \begin{pmatrix} 1 & T_s \\ 0 & 1 \end{pmatrix} \mathbf{x}(t) + \begin{pmatrix} T_s^2/2 \\ T_s \end{pmatrix} u(t) \tag{8}$$

where $T_s$ is the sample time, and the batch vector

$$\overline{\mathbf{x}} = (p(1)\ v(1)\ p(2)\ v(2)\ \cdots\ p(N)\ v(N))^{\mathsf{T}}. \tag{9}$$

The system is not controllable since $n_u < n_x$. To explain this, consider the dynamics and assume that at time $t$ the position $p(t) = a$ and the velocity $v(t) = b$ for some constants $a$ and $b$. It should be possible to choose the position and velocity at the next time step $t+1$ arbitrary to have controllability for the system in the iteration domain. It can be noticed from (8) that it is impossible to go from $p(t) = a$ and $v(t) = b$ to an arbitrary point at time $t+1$, hence $\overline{\mathbf{x}}$ cannot be chosen arbitrary.

If the position or the velocity is considered as output, then the necessary condition for output controllability is satisfied. It turns out that the system is output controllable in both cases by examining the rank of the matrix $\boldsymbol{\mathcal{C}}\mathbf{S}_{\mathbf{xu}}$.

If instead Euler sampling is used, then the discrete-time model becomes

$$\mathbf{x}(t+1) = \begin{pmatrix} 1 & T_s \\ 0 & 1 \end{pmatrix} \mathbf{x}(t) + \begin{pmatrix} 0 \\ T_s \end{pmatrix} u(t). \tag{10}$$

By considering the position as output gives that the first row in $\boldsymbol{\mathcal{C}}\mathbf{S}_{\mathbf{xu}}$ is equal to zero because of the zero element in $\mathbf{B}_{\mathbf{u}}$, hence the rank condition for $\boldsymbol{\mathcal{C}}\mathbf{S}_{\mathbf{xu}}$ is not satisfied. It means that the control signal does not affect the position directly and it follows that system (3) can require up to $n_x$ time steps before it can reach the reference trajectory.

## 5. TARGET PATH CONTROLLABILITY

Output controllability concerns the possibility to reach a desired output at a specific time. For ILC it is of interest to reach a desired trajectory, in as few steps as possible, and then be able to follow that trajectory, hence it is more interesting to use the concept of target path controllability (TPC) [Engwerda, 1988]. Target path controllability with lead $p$ and lag $q$ will be abbreviated as TPC$(p, q)$.

In Engwerda [1988] several results are presented to guarantee a system to be TPC. Theorem 3 states a requirement for the system in (3) to be TPC$(p, q)$.

*Theorem 3.* A linear time-invariant system is TPC$(p, q)$ if and only if rank $\mathbf{S}_{\mathbf{yu}}(p, q) = qn_y$, where

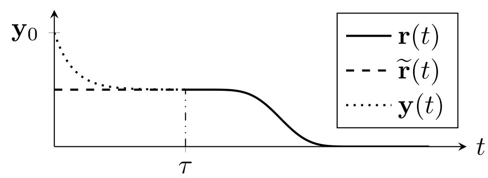


Fig. 1. Augmentation of the reference trajectory to include lead-in.

$$\mathbf{S}_{\mathbf{yu}}(p, q) = \begin{pmatrix} \mathbf{CA}^{p-1}\mathbf{B}_{\mathbf{u}} & \cdots & \mathbf{CB}_{\mathbf{u}} & \cdots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ \mathbf{CA}^{p+q-2}\mathbf{B}_{\mathbf{u}} & & \cdots & & \mathbf{CB}_{\mathbf{u}} \end{pmatrix}.$$

The connection between TPC and output controllability is presented in Theorem 4

*Theorem 4.* Output controllability of the system in the iteration domain is equivalent to the system in (3) being TPC$(1, N)$.

Return to the example in Section 4 for the case where Euler sampling has been used and the position is the output. Removing the first row with only zeros in $\boldsymbol{\mathcal{C}}\mathbf{S}_{\mathbf{xu}}$ gives the matrix $\mathbf{S}_{\mathbf{yu}}(2, N-1)$. The condition in Theorem 3 are now satisfied, hence the system is TPC with lead 2.

## 6. CONCEPT OF LEAD-IN

Target path controllability can now be used to investigate after how many samples it is possible to track the reference, and during how many samples the reference can be tracked. It comes now naturally to define the concept of lead-in. Lead-in means that the starting point of the original reference trajectory $\mathbf{r}(t)$ is moved $\tau$ samples forward in time by appending the reference with a new initial part $\widetilde{\mathbf{r}}(t)$, see Figure 1. The output now follows the new reference signal. The assumption of the system being TPC with lead $p \le \tau$ means that the system should be able to follow the original reference $\mathbf{r}(t)$. The error in the beginning, i.e., $\widetilde{\mathbf{r}}(t) - \mathbf{y}(t)$ for $t \le \tau$, does not matter since the aim is to follow $\mathbf{r}(t)$. Note that lead-in may not always be possible to use in practice. If the application and the trajectory do not permit to append $\widetilde{\mathbf{r}}(t)$, then lead-in cannot be used.

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