Homework set # 3 for the course
"Opinion Dynamics on Social Networks"

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1. **Consensus type.** Which of the following adjacency matrices

$$A_1 = \begin{bmatrix} 0 & 0 & 0.8 & 0.9 & 0 \\ 0 & 0 & 0.9 & 0 & 0.8 \\ 0.8 & 0.9 & 0 & 0 & -0.3 \\ 0.9 & 0 & 0 & 0 & 0 \\ 0 & 0.8 & -0.3 & 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 0.7 & 0 & 0.2 & 0 \\ 0 & 0.1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0.5 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0.4 \\ 1.2 & 0.3 & 0 & 1.5 & 0 \\ 0 & 0 & 0.7 & 0 & 0 \\ 0 & 0 & 0.2 & 0 & 1.3 \\ 2.1 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad A_4 = \begin{bmatrix} 0 & 0 & -0.6 & 0 & 0 \\ 0.2 & 0.4 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -0.3 & 0.8 & 1.2 & 0 \\ -1.1 & -0.5 & 0 & 0 & 0 \end{bmatrix}$$

$$A_5 = \begin{bmatrix} 0 & 0 & -0.6 & 0 & 0 \\ 0.2 & 0.4 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -0.3 & 0.8 & 1.2 & 0 \\ 1.1 & -0.5 & 0 & 0 & 0 \end{bmatrix}$$

can achieve consensus (in continuous-time) and to what type (average, weighted average, bipartite). Motivate your answer theoretically. Provide also a simulation result to verify your answer.

2. **Multidimensional Friedkin-Johnsen (FJ) model.** Recall the discrete-time FJ model discussed in class:

$$x(t+1) = (I - \Theta)Wx(t) + \Theta x(0) \tag{1}$$

where $x \in \mathbb{R}^n$ is the opinion vector ($n =$ n. of agents), W is a row stochastic matrix of interactions, and $\Theta = \text{diag}(\theta_1, \ldots, \theta_n)$, with $\theta_i \in [0,\, 1)$ the stubbornness coefficients.

Assume that we have m issues to discuss, and that we want to discuss them simultaneously (not concatenated!) because they are interdependent. Each of them is modeled as an FJ model, but these models must be run simultaneously because they are interdependent. Each agent has now a vector of m opinion variables: $x_i = \begin{bmatrix} x_i^1 & \ldots & x_i^m \end{bmatrix}^\top \in \mathbb{R}^m$.

Denoting $C \in \mathbb{R}^{m \times m}$ the topic interdependence matrix, then a possible model is the following

$$\bar{x}(t+1) = (I - \Theta)W \otimes C\bar{x}(t) + \Theta \otimes I_m \bar{x}(0) \tag{2}$$

where $\otimes =$ Kronecker product and $\bar{x} = \begin{bmatrix} x_1^\top & \dots & x_n^\top \end{bmatrix}^\top \in \mathbb{R}^{nm}$.

(a) For the case $m = 2$, simulate the model using the following matrices:

$$W = \begin{bmatrix} 0.22 & 0.12 & 0.36 & 0.3 \\ 0.147 & 0.215 & 0.344 & 0.294 \\ 0 & 0 & 1 & 0 \\ 0.09 & 0.178 & 0.446 & 0.286 \end{bmatrix} \quad \text{and} \quad \Theta = \operatorname{diag}(W)$$

and using as initial condition

$$x_1(0) = \begin{bmatrix} 25 \\ 25 \end{bmatrix}, \quad x_2(0) = \begin{bmatrix} 25 \\ 15 \end{bmatrix}, \quad x_3(0) = \begin{bmatrix} 75 \\ -50 \end{bmatrix}, \quad x_4(0) = \begin{bmatrix} 85 \\ 5 \end{bmatrix}$$

What happens for the following interdependence matrices? (report the final state, or a simulation plot)

$$C_1 = I, \quad C_2 = \begin{bmatrix} 0.8 & 0.2 \\ 0.3 & 0.7 \end{bmatrix}, \quad C_3 = \begin{bmatrix} 0.8 & -0.2 \\ -0.3 & 0.7 \end{bmatrix}$$

(b) Show that if C is row (sub)stochastic[1] then the evolution stays in the convex hull: $x_i(0) \in [a, b]^m$ for all $i \implies x_i(t) \in [a, b]^m$ for all i, where $[a, b]$ is an interval of $\mathbb{R}$.

(c) Show that the model is Schur stable[2] if and only if $\rho(C) < \rho((I - \Theta)W)^{-1}$.

(d) Show that when $\rho(C) = 1$ then the model (2) is Schur stable if and only if the model (1) is Schur stable.

[1] C is substochastic if $C \geq 0$ and $C\mathbb{1} \leq 1$.

[2] A matrix is Schur stable if all its eigenvalues are in the open unit disk.