Motion on submanifolds of noninvariant holonomic constraints for a kinematic control system evolving on a matrix Lie group

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Abstract

For a control system on a matrix Lie group with one or more configuration constraints that are not left/right invariant, finding the combinations of (kinematic) control inputs satisfying the motion constraints is not a trivial problem. Two methods, one coordinate-dependent and the other coordinate-free are suggested. The first is based on the Wei-Norman formula; the second on the calculation of the annihilator of the coadjoint action of the constraint one-form at each point of the group manifold. The results are applied to a control system on SE(3) with a holonomic inertial constraint involving the noncommutative part in a nontrivial way. The difference in terms of compactness of the result between the two methods is considerable.

Keywords: matrix Lie groups, constrained motion, Wei-Norman formula, noninvariant one-forms, coadjoint action.

1 Introduction

The purpose of this paper is to describe the kinematic equations of a control system whose configuration space is a Lie group with one or more holonomic constraints. Unlike the usual first order nonholonomic constraints that originate from underactuation in kinematic control systems with body-fixed actuators, the constraints we treat here are expressed as algebraic relations between the state variables and can for example describe a particular control task or a requirement that has to be fulfilled when doing motion planning. If both a control system and a constraint are invariant, it is possible to confine the system to a subgroup of codimension equal to the dimension of the constraint and to drop the constraint itself from the problem formulation. This is the case normally treated in the literature, see for example [6]. In this letter we are instead interested in constraints which are neither left nor right invariant and for which the "quotienting out" of the constraint is not possible. Our main purpose is to propose two different methods to treat such a case.

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Consider an actuated rigid body. If the configuration constraints are inertial, i.e., admit a natural description in an inertial frame for all times, the spontaneous way to represent them is to consider canonical coordinates of the second kind, i.e., based on the product of exponentials formula [12]. In fact, in this case, provided a suitable basis is chosen, a constraint can be directly inserted in place of the corresponding state, without locally affecting the remaining configuration variables. On the other hand, when the actuators of a kinematic control system are fixed on the body, the dynamics is better described in terms of single exponential representation, i.e., of one-parameter flow of a single time-varying vector field. In order to map the constraint into the space of inputs, the Wei-Norman formula [13] can be used. Such formula is a local diffeomorphism, which is global for solvable groups like $SE(2)$, but not for more complicated groups like $SE(3)$. The main problem with this type of formulae is that it is coordinate-dependent and, for nonabelian groups (and nonabelian constrained submanifolds), the representation of the constraint in the single exponential is not unique but at each point of the group depends on the path chosen to leave that point. So it does not give a clear idea of what the constraints look like in the input space. In other words, while locally the first order contribution is clear in both systems of canonical coordinates, due to the noncommutativity, the Wei-Norman formula gives only a "slice" of the admissible input combinations that satisfy the constraints to the higher order terms.

In order to describe the entire annihilator space of a constraint at a point g of the group, we compute all the directions that satisfy the constraint by evaluating the corresponding vector fields at the identity. In fact, by assumption, the one form describing the constraint is not invariant, so it will look different on different points of the group. The adjoint map allows us to Lie algebra evaluate the control vector fields at any g in the reachable space. Since we have body fixed actuators (and therefore left invariant control vector fields), it is reasonable to pull the one form of the constraint back to the identity, pairing it with the adjoint of the admissible tangent directions at g. For the sake of simplicity we will work under the assumption of full actuation, although even the weaker assumption of controllability would have sufficed. Using standard pairing, the problem of finding the orthogonal subspace to the constraint at g is transformed into the problem of annihilating the coadjoint action on the one-form representing the constraint. All the combinations of inputs that satisfy the constraint at g can then be parameterized in terms of the components of (the matrix representation) of g. We would like to emphasize that these components are still coordinate-independent. Obviously, the explicit calculation of the input functions has to pass through a coordinatization of the group manifold, but, unlike in the Wei-Norman case, the independence from the path followed by the flow of the system is retained. The main advantage is that, no matter how complicated a constraint may look, at each point it is transformed into an algebraic equation linear in the inputs.

The example used throughout the paper is a system in $SE(3)$. The constraint here is that the roll in spatial coordinates must be kept constantly equal to zero. This corresponds locally to a one-from on the 3-dimensional space of rotations. Due to the noncommutativity of $SO(3)$, the one-form cannot be quotiented out i.e., cannot be globally expressed as a one-parameter curve in $SE(3)$ (nor in $SO(3)$).

2 Motion in presence of constraints

Consider a drift-free left-invariant kinematic control system evolving on a matrix Lie group G

$$\dot{g} = g \sum_{i=1}^{n} A_i u_i \qquad g(0) = g_0 \qquad g \in G \tag{1}$$

For sake of simplicity, we assume that the system is fully actuated i.e., $n = \dim G$. Notice that for all the considerations of this paper controllability would suffice; however, the formulae would be rather complicated. If the system (1) is fully actuated then $A_1, \ldots, A_n$ form a basis of the Lie algebra $\mathfrak{g}$ whose associate structure constants c_{ij}^k are defined by

$$[A_i,\, A_j] = \sum_{k=1}^{n} c_{ij}^k A_k \quad i,\, j = 1, \ldots, n \tag{2}$$

where, because of the skew-symmetry of the Lie bracket $c_{ij}^k = -c_{ji}^k$.

Here we follow [8], Appendix A. Assume we have a constraint on the trajectories allowed for our system:

$$\phi(g) = 0 \tag{3}$$

where $\phi : G \to \mathbb{R}$ and $g \in G$. Such a constraint is holonomic as it is imposed on the configuration space of the manifold. Differentiating, the constraint can be described in terms of 1-forms as

$$\langle \mathrm{d}\phi(g),\, \dot{g} \rangle = 0 \tag{4}$$

with $\mathrm{d}\phi(g) \in T_g^*G$, $\dot{g} \in T_gG$ and $\langle \cdot\,,\, \cdot \rangle$ is a standard nondegenerate pairing $T^*G \times TG \to \mathbb{R}$. Using left invariance, we obtain a Lie algebra evaluated pair

$$\begin{aligned} \langle \cdot\,,\, \cdot \rangle \;:\;\; & \mathfrak{g}^*,\; \mathfrak{g} && \to \mathbb{R} \\ & A_j^\flat,\; A_i && \mapsto \langle A_j^\flat,\, A_i \rangle = \delta_i^j \end{aligned} \tag{5}$$

that allows us to identify a basis $\left\{A_1^\flat,\, A_2^\flat, \ldots A_n^\flat\right\}$ of $\mathfrak{g}^*$. If, as is often the case, a (pseudo) Riemannian metric structure is chosen, then the pairing is given by a symmetric nondegenerate bilinear form, for example corresponding, for compact matrix groups, to the Killing form, i.e., to the trace of the matrix product of the adjoint representation:

$$\langle A_j^\flat,\, A_i \rangle = k_i^j \,\mathrm{tr}(\mathrm{ad}_{A_j} \mathrm{ad}_{A_i}) = \delta_i^j$$

where $k_i^i \neq 0$. The $A_j^\flat$, $j = 1, \ldots, n$, provide a basis $\left\{ {L_{g^{-1}}}_* A_1^\flat, \ldots, {L_{g^{-1}}}_* A_1^\flat \right\}$ on T_g^*G via the push forward map.

A constraint $\mathrm{d}\phi$ is said *left-invariant* if

$${L_g}^* \mathrm{d}\phi(g) = \mathrm{d}\phi(I), \qquad I = \text{identity of } G$$

i.e., if the pull-back of the constraint to the identity corresponds to the constraint at the identity. This is the case when the "path" followed by the constraint from the identity to g can be expressed as a one-parameter subgroup of G. In particular, $\phi(g) = 0$ is left/right invariant if and only if $\phi^{-1}(0)$ is a subgroup H of G (see Loncaric [6] for a formulation on $SE(3)$). In other words, for a left-invariant constraint, there exists a left-invariant distribution of feasible velocities (the *free* velocities) that corresponds to the annihilator of the constraint at each point and that constitutes a subgroup. In this case, H is left-invariant with respect to the constraint and it can be factored out; the system on G/H is not anymore constrained. Here however, we are interested in more complicated one-forms than the left-invariant (or, specularly, right-invariant) ones, i.e., we want to consider constraints whose pull-back to the Lie algebra ${L_g}^* \mathrm{d}\phi(g)$ depends on g.

3 A coordinate-dependent formulation: the Wei-Norman formula

A coordinate-dependent way to describe the orthogonal space of a constraint is to use the Wei-Norman lemma. There are two types of local representations of the solution of the system (1), called canonical coordinates of the first and second kind. They rely respectively on the single exponential representation (due to Magnus [7]) and on the product of exponentials formulation (due to Wei-Norman [13]). The understanding of the relation between the two formulations for a control system is first due to Brockett [1]. See also [4, 11] for recent uses of this idea for motion planning of underactuated systems on matrix Lie groups.

Let $g(t) \in G$ be the solution of the system (1) starting with initial condition $g(0) = g_0$. Then there exists a neighborhood of $t = 0$ in which $g(t)$ can be expressed as a product of exponentials

$$g(t) = g_0 e^{\gamma_1(t)A_1} e^{\gamma_2(t)A_2} \dots e^{\gamma_n(t)A_n} \tag{6}$$

The Wei-Norman coordinate functions $\gamma_i(t)$, $i = 1, \dots, n$, are scalar functions of t and evolve according to the set of differential equations on $\mathbb{R}^n$:

$$\begin{bmatrix} \dot\gamma_1(t) \\ \vdots \\ \dot\gamma_n(t) \end{bmatrix} = \Xi(\gamma_1(t), \dots, \gamma_n(t))^{-1} \begin{bmatrix} u_1(t) \\ \vdots \\ u_n(t) \end{bmatrix} \tag{7}$$

where $\gamma_i(0) = 0$ and the matrix $\Xi(\cdot)$ is a real analytic function of the γ_i.

If Ad_g is the adjoint map

$$\begin{aligned} \mathrm{Ad}_g \quad : \quad \mathfrak{g} \quad &\to \mathfrak{g} \\ A \quad &\mapsto \mathrm{Ad}_g A = gAg^{-1} \end{aligned} \tag{8}$$

the explicit calculation of $\Xi(\gamma(t))$ can be done comparing the expression (1) with the derivative of g with respect to the product of exponentials (6), (see [2]).

$$\begin{aligned} \dot g(t) &= g_0 \Big(\dot\gamma_1(t) A_1 e^{\gamma_1(t)A_1} e^{\gamma_2(t)A_2} \dots e^{\gamma_n(t)A_n} + \\ &\quad + \dot\gamma_2(t) e^{\gamma_1(t)A_1} A_2 e^{\gamma_2(t)A_2} \dots e^{\gamma_n(t)A_n} + \dots + \\ &\quad + \dot\gamma_n(t) e^{\gamma_1(t)A_1} \dots e^{\gamma_{n-1}(t)A_{n-1}} A_n e^{\gamma_n(t)A_n} \Big) \\ &= g(t) \Big(\dot\gamma_1(t) \mathrm{Ad}_{\left(\prod_{i=n}^{1} e^{-\gamma_i(t)A_i}\right)} A_1 + \\ &\quad + \dot\gamma_2(t) \mathrm{Ad}_{\left(\prod_{i=n}^{2} e^{-\gamma_i(t)A_i}\right)} A_2 + \dots + \\ &\quad + \dot\gamma_n(t) \mathrm{Ad}_{\left(e^{-\gamma_n(t)A_n}\right)} A_n \Big) \end{aligned}$$

The comparison has to be done along each of the basis elements of $\mathfrak{g}$, i.e., we need to compute the contribution of the adjoint operators

$$\mathrm{Ad}_{\left(\prod_{i=n}^{j} e^{-\gamma_i(t)A_i}\right)} A_j = \prod_{i=n}^{j} \left(e^{-\mathrm{ad}A_j \gamma(t)} \right) A_j \tag{9}$$

in terms of the A_i using the formula

$$e^{-B\gamma} A e^{B\gamma} = \sum_{k=0}^{\infty} \frac{(-1)^k \mathrm{ad}_B^k A}{k!} \gamma^k$$

where $\mathrm{ad}_B^0 A = A$, $\mathrm{ad}_B^1 A = [B, A]$ and $\mathrm{ad}_B^k A = \left[B, \mathrm{ad}_B^{k-1} A\right]$. Iterating this procedure j times, it is possible to write explicitly (9) in terms of the structure constants of the Lie algebra and therefore to obtain an explicit expression for the $\Xi(\gamma(t))$. The complete calculation can be found in the original paper [13] or in the book [10]. Notice the compatibility of the initial conditions in the two expressions for $\dot{g}(t)$, which implies that $\Xi(\gamma(0)) = I$ and therefore $\Xi(\cdot)$ locally invertible. Notice moreover that, by the left invariance, the initial state g_0 of the system does not appear in $\Xi(\gamma(t))$. If $\mathfrak{g}$ is solvable, then there exist coordinate functions γ_i that are globally valid, while this is not true for semisimple Lie algebras. In this case, the nonsingularity of Ξ has to be checked at the point of application. Using the expression (7), the constraint relation at g in terms of its coordinates $\gamma = (\gamma_1, \ldots, \gamma_n)$ becomes

$$\mathrm{d}\phi(\gamma(t))\dot{\gamma}(t) = \mathrm{d}\phi(\gamma)\Xi(\gamma(t))^{-1} u(t) = 0 \tag{10}$$

which is linear in the inputs and can be solved locally using the implicit function theorem. $\Xi(\cdot)$ changes according to the order chosen for the basis elements and therefore also the input combination solution of (10) looks different with different basis ordering. Only one possible combination of inputs that satisfies the constraints is captured at a time by the method just presented. This hides the geometric structure of the annihilator of $\mathrm{d}\phi(g)$.

4 A coordinate free formulation

The Wei-Norman types of formulae are intrinsically coordinate-dependent. In fact, as we saw above the configuration space admits different sets of canonical coordinates of the second kind, with equivalent properties. In particular, the different parameterizations represent different paths composed of one-parameter subgroups that allow to reach the same end-point g while satisfying (3). What we really want to compute is the set of velocities that satisfies the constraint, i.e., that leave the constraint invariant at a given point. As we have body fixed actuators, it is natural to work in body fixed frame considering the one-form $\mathrm{d}\phi$ as applied at $g \in G$, and pull it back to the identity. The Lie algebra evaluated constraint $\mathrm{d}\phi_g = L_g^* \mathrm{d}\phi(g)$ then leaves on $\mathfrak{g}^*$, the dual of the Lie algebra $\mathfrak{g}$.

For a fixed $g \in G$, $g \neq I$, we want to compute all the vector fields X such that $X(g) = \dot{g} = gX(I)$ satisfies the constraint. However, pairing directly $\mathrm{d}\phi_g$ and $X(I)$ (or, similarly, $\mathrm{d}\phi(g)$ and $X(g)$) just gives us the one-parameter solutions corresponding to the first order terms of the Taylor expansion, neglecting those due to noncommutativity. Therefore, the method we use here consists in "forgetting" about the way g is reached and considering only how much the tangent vector $\eta = X(g) \in T_g G$ differs from $\zeta = X(I) \in \mathfrak{g}$. In formulae, this is done by means of the adjoint map (9). The task is then to look for all $\zeta \in \mathfrak{g}$ such that

$$\langle \mathrm{d}\phi_g,\, \mathrm{Ad}_{g^{-1}}\zeta \rangle = 0 \tag{11}$$

$\mathrm{Ad}_{g^{-1}}$ gives an inner automorphism of the Lie algebra, i.e., a change of basis in $\mathfrak{g}$ corresponding to the composition of left and right invariant maps and (11) gives the subspace of $\mathfrak{g}$ annihilated by $\mathrm{d}\phi_g$ for a fixed g.

For a generic transformation connecting I to g (referred to I), the *coadjoint action* $\mathrm{Ad}_{g^{-1}}^* : \mathfrak{g}^* \to \mathfrak{g}^*$ is defined through the pairing (see [9])

$$\langle \mathrm{Ad}_{g^{-1}}^* \mu,\, \zeta \rangle = \langle \mu,\, \mathrm{Ad}_{g^{-1}}\zeta \rangle \qquad \mu \in \mathfrak{g}^*, \quad \zeta \in \mathfrak{g} \tag{12}$$

In our case, we can use (12) to reformulate the problem (11) in the following way:

$$\textit{For a fixed } g \in G, \textit{ find all } \zeta \in \mathfrak{g} \textit{ such that} \qquad \langle \mathrm{Ad}_{g^{-1}}^* \mathrm{d}\phi_g,\, \zeta \rangle = 0 \tag{13}$$

In the assumption of transitivity, g can assume any value in G. At $g \in G$, left invariance of the control system means that (1) can be rewritten as $\dot{g} = g\zeta$ where $\zeta = \sum_{i=1}^{n} A_i u_i$. Equation (13) then gives a linear system of algebraic equations with the inputs as unknowns and $\mathrm{Ad}^*_{g-1}\mathrm{d}\phi_g$ as coefficients. Its solution gives all the combinations of control inputs belonging to the annihilator of the constraint.

One may wonder what kind of structure such a space has. In order to see this, consider the constraint at the identity of G: $\mathrm{d}\phi(I) = \mathrm{d}\phi_I$. We can choose a basis $\{A_1, A_2, \dots A_n\}$ of $\mathfrak{g}$ adapted to the constraint, such that the constraint can be described in $\mathfrak{g}^*$ simply as

$$\langle \mathrm{d}\phi(I), \cdot\rangle = \langle A_i^\flat, \cdot\rangle = 0 \qquad \text{for a fixed } i$$

If a control system is fully actuated, then its control vector fields can be expressed in terms of any given basis, just by taking fictitious inputs that are linear combinations of the original controls. Therefore we can always think of (1) as already in the basis adapted to the constraint.

The submanifold $\mathrm{Orb}(A_i^\flat) = \left\{\mathrm{Ad}^*_{g^{-1}} A_i^\flat,\ g \in G\right\}$ is called the *coadjoint orbit* of G on $A_i^\flat$, it has a symplectic structure and it is diffeomorphic to $G/G_{A_i^\flat}$, where $G_{A_i^\flat}$ is the closed isotropy subgroup of the coadjoint action at $A_i^\flat$:

$$G_{A_i^\flat} = \left\{g \in G \mid \mathrm{Ad}^*_g A_i^\flat = A_i^\flat\right\} \tag{14}$$

Deriving the pairing (12) with respect to g, at the identity

$$\langle \mathrm{ad}^*_\xi \mu, \zeta\rangle = \langle \mu, \mathrm{ad}_\xi \zeta\rangle \qquad \mu \in \mathfrak{g}^*, \quad \zeta \in \mathfrak{g} \tag{15}$$

where $\xi = \frac{d}{dt} g(t)\big|_{t=0}$, the Lie algebra to $G_{A_i^\flat}$ is

$$\mathfrak{g}_{A_i^\flat} = \left\{A_j \in \mathfrak{g} \mid \mathrm{ad}^*_{A_j} A_i^\flat = 0\right\}$$

From (14), both the isotropy subgroup and its Lie algebra have dimension equal to the number of constraints and do not admit a left/right invariant representation. So the tangent algebra of the coadjoint orbit is the annihilator of $\mathfrak{g}_{A_i^\flat}$ in $\mathfrak{g}^*$:

$$\mathfrak{g}^\circ_{A_i^\flat} = \left\{\eta \in \mathfrak{g}^* \mid \langle \eta, A_j\rangle = 0 \ \ \forall A_j \in \mathfrak{g}_{A_i^\flat}\right\}$$

From the coadjoint orbit theorem (see [9] Thm.14.3.1), the symplectic 2-form on $\mathrm{Orb}(\mathrm{A}_\mathrm{i}^\flat)$ is given by

$$\omega(\nu)\left(\mathrm{ad}^*_\zeta(\nu),\ \mathrm{ad}^*_\rho(\nu)\right) = \langle \nu,\ [\zeta,\ \rho]\rangle \quad \nu \in \mathrm{Orb}(A_i^\flat), \quad \zeta,\ \rho \in \mathfrak{g}$$

5 Example on $SE(3)$

For sake of simplicity, we consider a constraint $\phi(g) = 0$ that corresponds to one of the coordinate directions, for example γ_i, being constant. If the constraint is linear, this does not imply a loss of generality, as a basis "adapted" to the constraint can always be chosen. So the constraint (4) can be expressed in terms of canonical coordinates of the second kind (6) along the flow of the system as $\dot{\gamma}_i = 0$. By considering the corresponding row in (7), this gives a constraint in the input space, i.e., a combination of inputs living in the annihilator of $\mathrm{d}\phi(g)$.

The example we investigate here is a system on the Special Euclidean group $SE(3)$. Such a group is the semidirect product of rotations and translations in 3-dimensional space i.e., $SE(3) =$

$SO(3) \circledS \mathbb{R}^3$. The task could be, for example, to move an autonomous vehicle (like a submarine vehicle [5] or an aerial vehicle [3]) keeping it "upright".

Choosing a body fixed frame on the system, the left invariant representation of the kinematic equations of motion is simply

$$\dot{g} = gV_b \qquad g(0) = g_0 \tag{16}$$

where $g \in SE(3)$ and $V_b \in \mathfrak{se}(3)$, the Lie algebra of $SE(3)$. In the homogeneous representation, a left invariant basis of $\mathfrak{se}(3)$ is given by the 6 matrices

$$A_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad A_3 = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$A_4 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad A_5 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad A_6 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

we can interpret A_1 as the infinitesimal generator of the roll angle (around the x axis), A_2 as the pitch generator and A_3 as the yaw generator (axis in the z direction). With such basis, the kinematic of the system (16) can be rewritten as in equation (1), where u_i are control signals produced by body-fixed actuators.

The most reasonable mathematical formulation of "upright", in the chosen basis, is in terms of a pair of constraints that leave only one rotational degree of freedom (yaw angle). When $u_1 = u_2 = 0$, the motion occurs around the vertical axis of rotation so that roll and pitch angles are kept constant. In this case, as $SO(3)$ is reduced to the 1-dimensional subgroup S^1 (i.e., the two-form constraint is left-invariant), the configuration space of the system reduces to the solvable group $SE(2) \times \mathbb{R}$, which is non commutative because of the semidirect action in $SE(2)$ but has trivial rotation part. As we are interested in what happens for noncommutative rotations, we do not consider this situation further.

The next best thing that a passenger on board of a vehicle can interpret as (noncommutative) upright ego-motion is probably to allow pitch motion but not roll rotation with respect to the body frame. In the canonical coordinates of the second kind, this situation is univocally described by the (holonomic) constraint

$$\dot{\gamma}_1(t) = 0 \tag{17}$$

5.1 Coordinate-dependent solution

Our task is now to characterize the motion on the submanifold that describes the constraint, more explicitly to transform the constraint (17) into constraints in the space of inputs of the system using the Wei-Norman formula. All the motion planning and/or stabilization will then have to be performed on the constrained input space. Once an ordering of the basis of $\mathfrak{se}(3)$ is chosen, the Wei-Norman formula (7) can be computed explicitly. For example, keeping the order given by the cardinality of the basis index, after long and tedious calculations we obtain the following formula:

$$\begin{bmatrix} u_1(t) \\ \vdots \\ \vdots \\ u_6(t) \end{bmatrix} = \begin{bmatrix} \cos\gamma_2\cos\gamma_3 & \sin\gamma_3 & 0 & 0 & 0 & 0 \\ -\cos\gamma_2\sin\gamma_3 & \cos\gamma_3 & 0 & 0 & 0 & 0 \\ \sin\gamma_2 & 0 & 1 & 0 & 0 & 0 \\ -\gamma_6\cos\gamma_2\sin\gamma_3 - \gamma_5\sin\gamma_2 & \gamma_6\cos\gamma_3 & -\gamma_5 & 1 & 0 & 0 \\ -\gamma_6\cos\gamma_2\cos\gamma_3 + \gamma_4\sin\gamma_2 & -\gamma_6\sin\gamma_3 & \gamma_4 & 0 & 1 & 0 \\ \gamma_5\cos\gamma_2\cos\gamma_3 + \gamma_4\cos\gamma_2\sin\gamma_3 & -\gamma_5\cos\gamma_3 + \gamma_5\sin\gamma_3 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot\gamma_1(t) \\ \vdots \\ \vdots \\ \dot\gamma_6(t) \end{bmatrix}$$

whose inverse around $\gamma(0) = 0$ is (compare with eg. [4])

$$\begin{bmatrix} \dot\gamma_1(t) \\ \vdots \\ \vdots \\ \dot\gamma_6(t) \end{bmatrix} = \begin{bmatrix} \sec\gamma_2\cos\gamma_3 & -\sec\gamma_2\sin\gamma_3 & 0 & 0 & 0 & 0 \\ \sin\gamma_3 & \cos\gamma_3 & 0 & 0 & 0 & 0 \\ -\tan\gamma_2\cos\gamma_3 & \tan\gamma_2\sin\gamma_3 & 1 & 0 & 0 & 0 \\ 0 & -\gamma_6 & \gamma_5 & 1 & 0 & 0 \\ \gamma_6 & 0 & -\gamma_4 & 0 & 1 & 0 \\ -\gamma_5 & \gamma_4 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_1(t) \\ \vdots \\ \vdots \\ u_6(t) \end{bmatrix} \tag{18}$$

The condition (17) reformulated in the input space is then:

$$u_1 = \tan\gamma_3 u_2 \tag{19}$$

A coordinate representation like the canonical coordinates of the second kind used here for the $SO(3)$ part makes use of a set of Euler angles. Beside being not global, (as can be seen in (18)), such local parameterization is also well-known for being non-unique because of the noncommutativity of the group. This fact reflects here in the non-uniqueness of the constraint we obtain in input space. In fact, restricting ourself to Euler parameter cases, the Wei-Norman formula depends on the order on which the basis elements are chosen. We can think of the computations above as corresponding to the triple ZYX of angles, i.e., rotation along A_1 followed by A_2 and then A_3. If instead we chose the opposite order XYZ, calling $\tilde\gamma_i$ the new local coordinates, the first line of the Wei-Norman formula looks like

$$\begin{bmatrix} \dot{\tilde\gamma}_1(t) \end{bmatrix} = \begin{bmatrix} 1 & \sin\tilde\gamma_1\tan\tilde\gamma_2 & \cos\tilde\gamma_1\tan\tilde\gamma_2 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_1(t) \\ \vdots \\ u_6(t) \end{bmatrix}$$

so that the constraint $\dot{\tilde\gamma}_1(t) = 0$ is transformed to

$$u_1 + \sin\tilde\gamma_1\tan\tilde\gamma_2 u_2 + \cos\tilde\gamma_1\tan\tilde\gamma_2 u_3 = 0 \tag{20}$$

Notice, in particular, that the control authority involved in the constraint varies in the two sets of local coordinates.

5.2 Coordinate-free solution

As before, we consider the constraint of zero roll angle (17). A point $g \in SE(3)$ is isomorphic to the pair $(R, p) \in SO(3) \times \mathbb{R}^3$. The adjoint map at g can be expressed as the 6×6 matrix

$$\mathrm{Ad}_g = \begin{bmatrix} R & 0 \\ \widehat{p}R & R \end{bmatrix} \tag{21}$$

where $\widehat{p}R = p \times R$. If we identify $\mathfrak{se}(3)$ with $\mathbb{R}^6$ and the basis A_i with the standard Euclidean basis e_i, then also $\mathfrak{se}(3)^* \simeq \mathbb{R}^6$, $A_i^\flat = \mathrm{e}_i$ and the matrix expression for the coadjoint $\mathrm{Ad}^*_{g^{-1}}$ is the transpose of $\mathrm{Ad}_{g^{-1}}$

$$\mathrm{Ad}^*_{g^{-1}} = \begin{bmatrix} R & \widehat{p}R \\ 0 & R \end{bmatrix} \tag{22}$$

so that the coadjoint action on $A_1^\flat \simeq \mathrm{e}_1$ is

$$\mathrm{Ad}^*_{g^{-1}} A_1^\flat = \begin{bmatrix} R & \widehat{p}R \\ 0 & R \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} r_{11} \\ r_{21} \\ r_{31} \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Using the inner product pairing in $\mathbb{R}^6$, the relation (13) reduces to:

$$r_{11}u_1 + r_{21}u_2 + r_{31}u_3 = 0 \tag{23}$$
$$r_{11}^2 + r_{21}^2 + r_{31}^2 = 1 \tag{24}$$

where $u = \{u_1, \dots, u_6\} \in \mathbb{R}^6 \simeq \mathfrak{se}(3)$ (or, in (13), $\zeta = \sum_{i=1}^6 A_i u_i$). As noticed above, only the orthogonal part of $SE(3)$ is involved in the constraint. The submanifold described by (23)-(24) is a two-sphere at each point $R \in SO(3)$, expressed in terms of the components of R, i.e., it depends on how the point R looks like in the group manifold (but still it is independent of any choice of local coordinates). Inserting (24) into (23):

$$r_{11}u_1 + r_{21}u_2 + u_3\sqrt{1 - r_{11}^2 - r_{21}^2} = 0 \tag{25}$$

While it is not so obvious to realize that (19) and (20) live in the same two-sphere, it is straightforward to check that indeed they satisfy (25) (up to a scale factor).

This two-sphere moves around on $T_gSO(3)$ according to the symplectic 2-form induced by the coadjoint orbit.

6 Conclusion

A coordinate-dependent and a coordinate-free description of noninvariant inertial configuration constraints for a kinematic control system on a matrix Lie groups are discussed in this paper. The use of the coadjoint action allows to transform the constraints at each point into algebraic equations which are linear in the inputs and are valid globally, while the solution based on the Wei-Norman formula is more cumbersome and might be affected by singularities.

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