Homogeneous Polynomial Forms for Simultaneous Stabilizability of Families of Linear Control Systems: a Tensor Product Approach

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Abstract

The paper uses the formalism of tensor products in order to deal with the problem of simultaneous stabilizability of a family of linear control systems by means of Lyapunov functions which are homogeneous polynomial forms. While the feedback synthesis seems to be nonconvex, the simultaneous stability by means of homogeneous polynomial forms of the uncontrollable modes yields (convex) necessary but not sufficient conditions for simultaneous stabilizability.

I. Introduction

For a linear system without inputs, an ℓ-time tensor product of its n-dimensional state space yields a polynomial system homogeneous of degree ℓ. See [5] for a survey of the use of tensor products (or Kronecker products) in systems and control theory, and [2], [8], [22], [25] for a few more recent applications in the robust stability analysis and robust control literature. The presentation as a "larger" linear system provided by a tensor product suggests the introduction of homogeneous polynomial forms of degree 2ℓ (hereafter denoted as 2ℓ-HP forms) as "quadratic forms" of the tensored state space, represented by symmetric matrices of dimension n^ℓ. These are of importance when the *simultaneous stability* of a family of plants is investigated using a Common Lyapunov Function (CLF) for the entire family [4]. Such family may be thought of as the vertexes of a matrix polytope as in the robust stability problem, see [1], or as the modes of a switching system, [16]. In this case, in fact, no sharp analytic criterion is known, only sufficient conditions based on convex algorithmic procedures. Since Quadratic CLF (QCLF) have the defect of giving conservative answers, various extensions have been investigated, like the use of parametric or piecewise quadratic functions [25], the use of Homogeneous Polynomial Lyapunov Functions (HPLF) [7], [24] or combinations of the two [2]. The search for a Common HPLF of degree 2ℓ (hereafter 2ℓ-HPCLF) in these papers is based on the use of the power transformation method, a variant of the tensor product introduced in the control literature by Brockett [6]. The use of HPLF is closely related to the recent efforts to use positive polynomials in the form of sum of squares [12], [14], [15], [17], [19], [21]. Adopting a tensor product or a power transformation basis leads to equivalent results in terms of existence of CLF.

A tensor product representation of a linear system is linear in the degree ℓ monomials, but it is *at the same time* multilinear in the original state vector. This coexistence of the two points of view makes the extension to a linear system with inputs straightforward, as it corresponds to adding extra terms with degree of homogeneity $\ell - 1$ in the state variables and linear in the input. The scope of this paper is to use the tensor product formalism to deal with the problem of *simultaneous stabilizability* by means of HP forms for families of linear systems with inputs. For a given pair (A, B), the uncontrollable subspace must contain only asymptotically stable modes and can be characterized in terms of convex cones in $\mathrm{Ker}B^T$ determined by the Lyapunov inequality [10]. If we have N pairs (A_i, B), in order to have simultaneous stabilizability and a CLF it is necessary that the corresponding N cones have nonnull intersection. For quadratic stabilizability, this is equivalent to the existence of QCLF as well as to solving the simultaneous synthesis problem [10]. This is not anymore true when quadratic forms are replaced by HP forms of order $2\ell > 2$. The condition of nonempty intersection of the 2ℓ-HPLF cones in $\mathrm{Ker}B^T$ turns out to be *necessary but not sufficient* for simultaneous stabilizability by means of 2ℓ-HPCLF. While this necessary condition for higher order stabilizability has a natural convex formulation, the same does not seem to be true for the computation of 2ℓ-HPLF. We provide one possible (nonconvex) way of formulating the synthesis problem via 2ℓ-HPLF. It gives an idea of the extra complications one encounters in this construction: the equation for the feedback gain is obtained in implicit form. Rendering it explicit (i.e, solving for the gain) implies imposing a number of bilinear constraints on the matrix representative of the HP form of order 2ℓ.

Unlike the free dynamics case, systems with inputs are not discussed in the literature in terms of power transformations: here the multilinearity of the tensors is crucial in determining how the “extended system” appears and is instrumental also in understanding the convexity in terms of 2ℓ-HP forms.

The state vector of an ℓ-time tensored system has dimension n^ℓ, hence its dimension grows exponentially with ℓ. However, due to the number of repeated monomials in a tensor, the true rate of growth of independent quantities is only binomial. For a vector, essentially a power transformation is a tensor product in which redundant entries are lumped together. Hence it contains all the information of a tensor product but in a reduced dimension. While on the one hand the use of tensor products highlights the multilinear structure and allows a deeper geometric insight into the problem under study (the necessary condition based on 2ℓ-HPLF cones being one example), on the other hand for the purposes of effective calculations it is convenient to resort to the corresponding dimension-reduced system given by the power transformation. Especially if one considers the fact that several software tools are becoming available that treat convex semidefinite programs for this last basis [20], [14], [13].

II. Basics on tensor products

For a complete survey of the properties of tensor products, we refer the reader to [11], [5]. Given the matrices $A = (a_{ij}) \in M_n = \mathbb{R}^{n\times n}$, $B = (b_{ij}) \in M_m$, consider the tensor product

$$A \otimes B = \begin{bmatrix} a_{11}B & \dots & a_{1n}B \\ \vdots & & \vdots \\ a_{n1}B & \dots & a_{nn}B \end{bmatrix} \in M_{nm} = \mathbb{R}^{nm\times nm}$$

and the tensor sum $(A \oplus B) = A \otimes I_m + I_n \otimes B \in M_{nm}$, where I_n is the identity matrix in M_n (the subindex n will be omitted when no confusion arises). Define a *permutation matrix* $\Phi(n, m) \in M_{nm}$ as

$$\Phi(n, m) = \sum_{i=1}^{n}\sum_{j=1}^{m} E_{ij} \otimes E_{ij}^T = \left[E_{ij}^T\right]_{\substack{1 \leqslant i \leqslant n \\ 1 \leqslant j \leqslant m}}$$

with $E_{ij} \in M_{n\times m} = \mathbb{R}^{n\times m}$ the elementary matrix having 1 in the $(i\,j)$-th slot and zero elsewhere. Then $\Phi(n, m)^T = \Phi(m, n)$. Using Φ: $B \otimes A = \Phi(n, m)\, A \otimes B\, \Phi(n, m)$. We will use the notation $\Phi_{h,\ell-h} = \Phi(n^h, n^{\ell-h})$ and $\tilde{\Phi} = (I \otimes \ldots \otimes I + \Phi_{1,\ell-1} + \Phi_{2,\ell-2} + \ldots + \Phi_{\ell-1,1})$. While $\Phi_{i,j}$ is invertible, $\tilde{\Phi}$ is not.

Denote $\bar{x}$ the ℓ-time tensor product of $x \in \mathbb{R}^n$ with itself: $\bar{x} = x \otimes x \otimes \ldots \otimes x \in \mathbb{R}^n \otimes \ldots \otimes \mathbb{R}^n = (\mathbb{R}^n)^{\otimes \ell}$. Given

$$\dot{x} = Ax, \qquad A \in M_n, \tag{1}$$

for $\bar{x}$ one gets

$$\dot{\bar{x}} = \bar{A}\bar{x}. \tag{2}$$

where $\bar{A} = A \oplus \ldots \oplus A \in M_n^{\otimes \ell}$. The extension of the tensor product formalism to systems with linear inputs is straightforward and results in ℓ terms of homogeneity $\ell - 1$ in x, one for each possible slot of the control. If $u \in \mathbb{R}^m$, given the linear control system

$$\dot{x} = Ax + Bu, \qquad A \in M_n, \quad B \in M_{n \times m}, \tag{3}$$

then from (2) we obtain for $\bar{x}$:

$$\begin{aligned} \dot{\bar{x}} =& \bar{A}\bar{x} + (B \otimes I \ldots \otimes I)\, u \otimes x \otimes \ldots \otimes x \\ & + \ldots + (I \ldots \otimes I \otimes B)\, x \otimes \ldots \otimes x \otimes u \end{aligned} \tag{4}$$

where $u \otimes x \otimes \ldots \otimes x \in \mathbb{R}^m \otimes (\mathbb{R}^n)^{\otimes(\ell-1)}$ and likewise for the other terms. In (4), the linear feedback $u = -Kx$ fits in a natural way, as it restores the homogeneity of order ℓ in x in the tensored ODE. The closed loop system is simply

$$\dot{\bar{x}} = (A - BK) \oplus \ldots \oplus (A - BK)\, \bar{x} = \overline{(A - BK)}\bar{x}. \tag{5}$$

The HP forms used in this paper are functions $V : (\mathbb{R}^n)^{\otimes \ell} \to \mathbb{R}$ of the tensored variable $\bar{x}$. They are always of even degree, as we make extensive use of "quadratic" type of representations $V(\bar{x}) = \bar{x}^T P \bar{x}$ emphasizing the matrix representation $P \in M_{n^\ell}$. In particular, when $P = P^T > 0$ then $V(\bar{x})$ is a sum of squares and can be treated as a convex function in $\bar{x}$. From $\frac{dV(\bar{x})}{dt} = \frac{\partial V(\bar{x})}{\partial x}\dot{x} = \frac{\partial V(\bar{x})}{\partial \bar{x}}\dot{\bar{x}}$, the total derivative of $V(\bar{x}) = \bar{x}^T P \bar{x}$ along (1) can be written as

$$\dot{V}(\bar{x}) = \bar{x}^T \left(\bar{A}^T P + P\bar{A}\right) \bar{x}. \tag{6}$$

III. Simultaneous Stabilization by HPCLF

In the first two Subsections, single mode stabilizability and feedback synthesis via HPLF will be discussed. HPCLF for simultaneous stabilizability will be treated in the third Subsection.

A. Stabilizability via HPLF

For the linear control system (3), stabilizability is well-known to be achievable only when the uncontrollable modes are already asymptotically stable, i.e., when the modes of (3) belonging to the nullspace of B^T are asymptotically stable. In fact, if $u = -Kx$ the Lyapunov inequality

$$(A - BK)\, W_q + W_q\, (A - BK)^T < 0, \tag{7}$$

$W_q \in M_n$, $W_q = W_q^T > 0$, reduces to

$$x^T \left(AW_q + W_q A^T\right) x < 0 \qquad \forall\, x \text{ s.t. } B^T x = 0, \tag{8}$$

i.e., to the cone

$$\mathcal{C}_A^2 = \big\{ W_q \in M_n \text{ s.t. } W_q = W_q^T > 0 \text{ and } x^T\big(AW_q + W_qA^T\big)\, x < 0, \\ \forall\, x \in \mathbb{R}^n \text{ s.t. } \|x\| = 1 \text{ and } B^T x = 0 \big\}\,.$$

Hence the system (3) is stabilizable if and only if the convex cone of $W_q = W_q^T > 0$ that satisfy (8) is nonempty. This is Theorem 1 in [10]. Expressed as an LMI, (8) is equivalent to the feasibility of

$$\left(B^{\perp}\right)^T \left(AW_q + W_qA^T\right) B^{\perp} < 0, \quad W_q = W_q^T > 0, \tag{9}$$

where $B^{\perp}$ is the orthogonal complement of B, i.e., $\left(B^{\perp}\right)^T B = 0$, $\mathrm{rank}[B^{\perp}\ B] = n$, see [3], §7.2.1.

Although no improvement can be expected for (3), we want now to formulate the same problem for a convex cone of HP forms of order 2ℓ, $\ell > 1$. The key observation is that for the tensored system (4) with the linear feedback $u = -Kx$, the condition that the modes in the nullspace of B^T must be asymptotically stable is still on place, while we can replace (8) with a larger LMI corresponding to $W \in M_{n^\ell}$. To understand this, it is enough to observe that $\bar{x}$ contains ℓ copies of the state vector x and that

$$\begin{aligned} B^T x = 0 \Longleftrightarrow\ & B^T \otimes I \otimes \ldots \otimes I\ \bar{x} = 0 \\ & \vdots \\ \Longleftrightarrow\ & I \otimes \ldots \otimes I \otimes B^T\ \bar{x} = 0. \end{aligned} \tag{10}$$

Hence, whenever $B^T x = 0$ the Lyapunov inequality

$$\overline{(A - BK)}W + W\overline{(A - BK)}^T < 0, \tag{11}$$

$W \in M_{n^\ell}$, $W = W^T > 0$, reduces to

$$\bar{x}^T \left(\bar{A}W + W\bar{A}^T\right) \bar{x} < 0 \qquad \forall\, x \ \text{ s. t. } \quad B^T x = 0, \tag{12}$$

where now we can search for W on the larger convex cone:

$$\mathcal{C}_A^{2\ell} = \big\{ W \in M_{n^\ell} \text{ s.t. } W = W^T > 0 \text{ and } \bar{x}^T\big(\bar{A}W + W\bar{A}^T\big)\, \bar{x} < 0, \\ \forall\, x \in \mathbb{R}^n \text{ s.t. } \|x\| = 1,\ B^T x = 0 \big\}\,.$$

Clearly (8) and (12) admit solutions simultaneously. In fact, $\mathrm{Ker} B^T$ is the same and $\bar{A}$ is stable if and only if A is.

An LMI test for nonemptiness of $\mathcal{C}_A^{2\ell}$ similar to (9) is:

$$\left(B^{\perp} \otimes \ldots \otimes B^{\perp}\right)^T \left(\bar{A}W + W\bar{A}^T\right) \left(B^{\perp} \otimes \ldots \otimes B^{\perp}\right) < 0, \tag{13}$$

$W = W^T > 0$. However, unlike the standard case, we do not have an "if and only if" condition between (11) and (13): while nonemptiness of $\mathcal{C}_A^2$ corresponds to nonemptiness of $\mathcal{C}_A^{2\ell}$, not all elements of $\mathcal{C}_A^{2\ell}$ correspond to feedback controllers and hence satisfy (11).

Putting together all these results we have the following generalization of Theorem 1 of [10].

Proposition 1 *For the system* (3)*, the following facts are equivalent:*

1) (A, B) *stabilizable;*
2) $\mathcal{C}_A^2$ *nonempty;*
3) $\mathcal{C}_A^{2\ell}$ *nonempty* $\forall\, \ell \in \mathbb{N}$*;*
4) *the LMI* (13) *is feasible* $\forall\, \ell \in \mathbb{N}$.

It is important to stress that (13) comes as a natural LMI formulation of $\mathcal{C}_A^{2\ell}$, since it guarantees that all terms containing B are canceled. Using a larger orthogonal complement would not specify $\mathcal{C}_A^{2\ell}$ any better.

B. Stabilization via HPLF

Recall that the standard solution to the feedback stabilization of (3) based on QLF provides the controller $K = R^{-1}B^TP_q$ for some $P_q \in M_n$, $P_q = P_q^T = W_q^{-1} > 0$, which also corresponds to the optimal solution of some quadratic cost functional. In this case, one has the closed loop system $\dot{x} = \left(A - BR^{-1}B^TP_q\right)x$ with Lyapunov inequality

$$\left(A - BR^{-1}B^TP_q\right)^T P_q + P_q\left(A - BR^{-1}B^TP_q\right) < 0. \tag{14}$$

The following Proposition provides an explicit construction for a stabilizing feedback based on Lyapunov functions which are HP of order 2ℓ. Consider the matrix $\tilde{P} = P\tilde{\Phi} \in M_{n^\ell}$, where $P \in M_{n^\ell}$, $P = P^T > 0$, P such that $P^{-1} = W \in \mathcal{C}_A^{2\ell}$. Partition $\tilde{P}$ into blocks of dimensions $n \times n$, $\tilde{P} = \left[\tilde{P}_{ij}\right]_{1\leqslant i,j\leqslant n^{\ell-1}}$, and impose on it the following linear constraints:

$$P \quad \text{s.t.} \quad \begin{cases} \tilde{P}_{ii}B = \tilde{P}_{jj}B & \forall\, 1 \leqslant i,\, j \leqslant n^{\ell-1} \\ \tilde{P}_{ij}B = 0 & \forall\, 1 \leqslant i,\, j \leqslant n^{\ell-1},\ i \neq j \end{cases}. \tag{15}$$

Notice that $\tilde{P}^T \neq \tilde{P}$ but that $\tilde{P}^T = \tilde{\Phi}P$. A matrix $P > 0$ satisfying (15) not always exists, depending on B. When it does, we can use it to construct a feedback stabilizer.

Proposition 2 *Assume* $(A,\, B)$ *stabilizable. Then, for some* $R \in M_m$, $R = R^T > 0$, *if* $\exists\, P \in M_{n^\ell}$, $P = P^T > 0$, *such that* $P^{-1} = W \in \mathcal{C}_A^{2\ell}$ *and* $\tilde{P}$ *obeys* (15), *then the control* $u = Kx$ *with* $K = R^{-1}B^T\tilde{P}_{11}^T$ *is a stabilizing linear state feedback law for* (3).

Proof The statement is true if the feedback gain $K = R^{-1}B^T\tilde{P}_{11}^T$ is such that the Lyapunov operator for the closed loop system tensored into $\overline{(A-BK)}^T P + P\overline{(A-BK)}$ is negative definite for some $P \in M_{n^\ell}$, $P = P^T > 0$, i.e., $\forall\, \bar{y} \in (\mathbb{R}^n)^{\otimes \ell}$, $\bar{y} \neq 0$

$$\bar{y}^T\left(\overline{(A-BK)}^T P + P\overline{(A-BK)}\right)\bar{y} < 0. \tag{16}$$

Using the permutation operators $\Phi_{h,\ell-h}$ and the identities $\Phi_{h,\ell-h}\bar{y} = \bar{y}$ and $\bar{y}^T\Phi_{h,\ell-h} = \bar{y}^T$, from (15) one gets:

$$\begin{aligned}
&\bar{y}^T\left(\overline{(A-BR^{-1}B^T\tilde{P}_{11}^T)}^T P + P\overline{(A-BR^{-1}B^T\tilde{P}_{11}^T)}\right)\bar{y} \\
&= \bar{y}^T\left(\bar{A}^T P + P\bar{A} - \left(\tilde{P}(I\otimes\ldots\otimes I\otimes BR^{-1}B^T)\tilde{P}^T\right)^T\right. \\
&\qquad \left. -\tilde{P}(I\otimes\ldots\otimes I\otimes BR^{-1}B^T)\tilde{P}^T\right)\bar{y}.
\end{aligned} \tag{17}$$

If $W = P^{-1}$ and $\bar{x} = W^{-1}\bar{y}$, then (17) becomes:

$$\bar{x}^T\left(W\bar{A}^T + \bar{A}W - 2\tilde{\Phi}\left(I\otimes\ldots\otimes I\otimes BR^{-1}B^T\right)\tilde{\Phi}\right)\bar{x}. \tag{18}$$

Now the $\tilde{\Phi}$ are irrelevant since $\tilde{\Phi}\bar{x} = \ell\bar{x}$. When restricting to $\mathcal{C}_A^{2\ell}$, $B^T x = 0$ and (12) holds. Then (18) is a particular instance of the Finsler's lemma (see [18], Lemma 4.1), implying that

$$W\bar{A}^T + \bar{A}W - 2\ell\left(I\otimes\ldots\otimes I\otimes BR^{-1}B^T\right) < 0 \tag{19}$$

for a suitable choice of R and for W such that P obeys (15). But if (19) holds then (16) holds, and the Proposition is proved. ∎

Rather than really constructive in practice, Proposition 2 is useful to understant the extra difficulties one encounters when dealing with HPLF. These are essentially due to the structured form (15) that HPLF have to have, in order to obtain $n^{\ell-1}$ copies of the same matrix feedback gain K, i.e., the same gain in all “copies” of the system in the tensored form:

$$
\begin{aligned}
I \otimes \ldots \otimes I \otimes K &= \begin{bmatrix} K & 0 & \ldots & 0 \\ 0 & K & & \\ \vdots & & \ddots & \\ 0 & & & K \end{bmatrix} \\
&= \begin{bmatrix} R^{-1}B^T\tilde{P}_{11}^T & 0 & \ldots & 0 \\ 0 & R^{-1}B^T\tilde{P}_{11}^T & & \\ \vdots & & \ddots & \\ 0 & & & R^{-1}B^T\tilde{P}_{11}^T \end{bmatrix} \\
&= \begin{bmatrix} R^{-1}B^T\tilde{P}_{11}^T & 0 & \ldots & 0 \\ 0 & R^{-1}B^T\tilde{P}_{22}^T & & \\ \vdots & & \ddots & \\ 0 & & & R^{-1}B^T\tilde{P}_{n^{\ell-1}n^{\ell-1}}^T \end{bmatrix}
\end{aligned}
$$

so that $\tilde{P}^T$ can be extracted from the right, yielding the rightmost expression in (18)-(19). The use of permutation matrices $\Phi_{h,\ell-h}$ is instrumental in obtaining an implicit form which is simple enough to be solvable for K with the addition of only linear constraints.

Notice that while for quadratic stabilizability it is possible to set up a convex LMI problem directly in (7) by simply replacing KW_q with a new matrix variable $S \in M_{m\times n}$ and studying the feasibility of the LMI

$$
AW_q + W_qA^T - BS - S^TB^T < 0, \tag{20}
$$

for the stabilizability with 2ℓ-HP forms this is not anymore possible. In fact, (11) reformulated as in (17) is a bilinear matrix inequality

$$
\begin{aligned}
&W\bar{A}^T + \bar{A}W - \left(\tilde{\Phi}\left(I \otimes \ldots \otimes I \otimes BK\right)W\right)^T \\
&- \tilde{\Phi}\left(I \otimes \ldots \otimes I \otimes BK\right)W < 0
\end{aligned}
$$

where it is not possible to lump together K and W without additional bilinear constraints, because of the block partitioned structure (15). If we call $(I \otimes \ldots \otimes I \otimes K)\,W = S$, $S \in M_{n^{\ell-1}m\times n^{\ell}}$, then one needs to carry along the bilinear constraints $KW_{ij} = S_{ij}$ where $W = [W_{ij}]_{1\leqslant i,\, j\leqslant n^{\ell-1}}$, $W_{ij} \in M_n$ and $S = [S_{ij}]_{1\leqslant i,\, j\leqslant n^{\ell-1}}$, $S_{ij} \in M_{m\times n}$. Instead in the formulation of Proposition 2 the constraints are just the linear constraints of (15).

Once the feedback law has been inserted into the system, stabilizability can be verified by means of the closed loop LMI (19) with $P = W^{-1}$ obeying (15). However, we do not have yet a tractable algorithm for the computation of P since (19) is in $W = P^{-1}$ while the constraints (15) are in P. Furthermore, the lack of invertibility of $\tilde{\Phi}$ means that it is not possible to re-express the constraints (15) in terms of W. Since $\bar{x} = P\bar{y}$, from (10), $I \otimes \ldots \otimes I \otimes B^T \bar{x} = 0$ holds if and only if $I \otimes \ldots \otimes I \otimes B^T P \bar{y} = 0$. Then to (17) the Finsler's lemma is also applicable, same as (18), yielding the ARE-like quadratic matrix inequality with linear constraints

$$\begin{aligned} \bar{A}^T P + P\bar{A} - 2\ell P\tilde{\Phi}(I \otimes \ldots \otimes I \otimes BR^{-1}B^T)\tilde{\Phi}P < 0, \\ P \text{ subject to (15)} \end{aligned} \tag{21}$$

or its more "balanced" counterpart

$$\bar{A}^T P + P\bar{A} - 2P\overline{(BR^{-1}B^T)}P < 0, \quad P \text{ subject to (15)} . \tag{22}$$

As is well-known, (21) (or (22)) is not convex.

C. Simultaneous stabilizability

In this work all plants have the same input matrix B while the state update matrix is $A(t) \in \mathrm{co}\{A_1, \ldots, A_N\}$. Quadratic stabilizability is verified in [10] by means of the convex cone

$$\mathcal{C}^2_{A_1,\ldots,A_N} = \left\{ W_q \in \mathcal{C}^2_{A_1} \cap \ldots \cap \mathcal{C}^2_{A_N} \right\} .$$

$\mathcal{C}^2_{A_1,\ldots,A_N}$ is nonempty when there exists a QCLF for $A_1, \ldots, A_N$ $\forall$ x such that $B^T x = 0$. The corresponding feedback controller is given for example by $K = R^{-1}B^T P_q$ where R is such that $R^{-1} \geqslant R_i^{-1}$, $i = 1, \ldots, N$ with R_i the solutions computed in $\mathcal{C}^2_{A_i}$ (recall that these where $K_i = R_i^{-1} B^T P_q$).

Likewise, for 2ℓ-HP forms one can test whether the cone

$$\mathcal{C}^{2\ell}_{A_1,\ldots,A_N} = \left\{ W \in \mathcal{C}^{2\ell}_{A_1} \cap \ldots \cap \mathcal{C}^{2\ell}_{A_N} \right\}$$

is nonempty, which corresponds to the simultaneous existence of a solution to N inequalities like (12): $\exists$ $W \in M_{n^\ell}$, $W = W^T > 0$ such that $\forall$ $i = 1, \ldots, N$

$$\bar{x}^T \left(\bar{A}_i W + W \bar{A}_i^T \right) \bar{x} < 0, \qquad \forall\, x \text{ s. t. } B^T x = 0. \tag{23}$$

Definition 1 *A family of linear systems defined by* $\{A_1, \ldots A_N\}$ *and* B *is*

- simultaneously quadratically stabilizable *via linear state feedback if* $\exists$ $W_q \in M_n$, $W_q = W_q^T > 0$ *and* $K \in M_{m\times n}$ *such that* $\forall$ $i = 1, \ldots, N$

$$(A_i - BK) W_q + W_q (A_i - BK)^T < 0; \tag{24}$$

- simultaneously stabilizable by 2ℓ-HPCLF *if* $\exists\, W \in M_{n^\ell}$, $W = W^T > 0$ *and* $K \in M_{m\times n}$ *such that* $\forall\, i = 1, \ldots, N$
$$\overline{(A_i - BK)}W + W\overline{(A_i - BK)}^T < 0. \tag{25}$$

The condition (23) is simply checked via N simultaneous feasibility problems like (13) i.e.,

$$\Big(B^\perp \otimes \ldots \otimes B^\perp\Big)^T \big(\bar{A}_i W + W \bar{A}_i^T\big) \Big(B^\perp \otimes \ldots \otimes B^\perp\Big) < 0, \tag{26}$$

$i = 1, \ldots, N$. The condition (26) (or (23)) is what makes the qualitative difference with respect to the quadratic case.

Proposition 3 *The condition* (26) *(or* (23)*) is a necessary but not sufficient condition for simultaneous stabilizability by* 2ℓ*-HPCLF of* $(\{A_1, \ldots, A_N\}, B)$.

Proof Necessity of (23) is obvious: if $\mathcal{C}^{2\ell}_{A_1,\ldots,A_N}$ is empty then (25) is also unfeasible. That (23) is not sufficient for 2ℓ-HPCLF stabilizability will be shown in Example 2. ■

When the set of P such that $\tilde{P} = P\tilde{\Phi}$ obeys (15), $P^{-1} \in \mathcal{C}^{2\ell}_{A_1,\ldots,A_N}$, is nontrivial, a controller simultaneously stabilizing all the plants is given by $K = R^{-1}B^T\tilde{P}_{11}^T$ with $R \in M_m$, $R = R^T > 0$, such that $R^{-1} \geq R_i^{-1}$, where the R_i are entering into the corresponding solutions computed from Proposition 2 for each (A_i, B). The proof follows directly from Proposition 2 and from the proof of Theorem 3 in [10].

Example 1 In correspondence of the interval state matrix and constant input matrix

$$A = \begin{bmatrix} [0.1,\, 1] & -1 & -1 \\ 2 & -1 & [-2,\, 1] \\ 1 & 1 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \tag{27}$$

the simultaneous stabilizability must be checked for 4 vertex pairs (A_i, B). Each $(A_i,\, B)$ is controllable, but it is not possible to bring all systems simultaneously into controllable canonical form. The quadratic test (24) gives a negative answer, while the test (26) with $\ell = 2$ provides a positive answer. Although Proposition 2 fails to determine a feedback law, in this particular example it is very easy to find (numerically) a simultaneously stabilizing linear gain, for example $K = \begin{bmatrix} -25 & -10 & 50 \end{bmatrix}$. This gain fails the test (25) with $\ell = 2,\, 3$ and W as unknown. Quite remarkably, so does *any* other numerically computed gain simultaneously stabilizing the A_i. ■

Example 2 Consider the pair of systems $(A_i,\, B)$, $i = 1,\, 2$.

$$A_1 = \begin{bmatrix} 10 & 0 & 0.1 \\ 1 & 0.3 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \tag{28}$$

Also in this case there is no QCLF, while the condition (26) with $\ell = 2$ is fulfilled. If the feedback gain is $K = \begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix}$, the characteristic polynomials of the two closed loop systems never fulfill simultaneously the Routh-Hurwitz stability criterion for any value of $k_j \in \mathbb{R}$, $j = 1,\, 2,\, 3$. Hence the two systems are never simultaneously stabilizable. ■

From the previous (and similar) examples, it is not clear if the constructive method proposed in Proposition 2 is more effective than quadratic synthesis. It is easily shown to be nonequivalent in the following problem of simultaneous stabilization by output feedback.

Example 3 (**simultaneous stabilization: state vs output feedback**) The following pair of systems ([23])

$$A_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -3 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \tag{29}$$

with output equation

$$z = Cx = \begin{bmatrix} 0 & -1 & 1 \end{bmatrix} x$$

is simultaneously quadratically stabilizable by state feedback, but not by (static) output feedback, i.e. $\nexists\, k_o \in \mathbb{R}$ and $W_q \in M_3$, $W_q = W_q^T > 0$ such that

$$(A_i - k_o BC)\, W_q + W_q\, (A_i - k_o BC)^T < 0, \quad i = 1,\, 2.$$

It is easily found (e.g. by a numerical search) that instead it is stabilizable by means of output feedback via a 4-HPCLF: $\exists$ a gain $k_o \in \mathbb{R}$ (e.g. $k_o = 0.1$) and a $W \in M_9$, $W = W^T > 0$ such that

$$\overline{(A_i - k_o BC)} W + W \overline{(A_i - k_o BC)}^T < 0, \quad i = 1,\, 2.$$

Notice that Proposition 3 remains valid also for output feedback stabilization and that (26) with $\ell = 2$ is satisfied. The solution can be obtained also by modifying the scheme developed in Section III-B. In this case one has to add an extra constraint: since $u = k_o z = k_o Cx = Kx$, the matrix P has to obey to yet another structure equation. In the case of a quadratic LF this is

$$k_o C = R^{-1} B^T P_q,$$

yelding

$$P_q = \begin{bmatrix} p_{11} & p_{12} & 0 \\ p_{12} & p_{22} & -p_{33} \\ 0 & -p_{33} & p_{33} \end{bmatrix}$$

which is never a CLF for the two closed loops. For the 4-HPCLF, instead, together with (15), the following must be imposed

$$k_o C = R^{-1} B^T \tilde{P}_{jj}^T, \quad j = 1,\, 2,\, 3,$$

yelding an extra number of linear constraints on $P \in M_9$. The structured problem one obtains in this way, however, admits a solution.

IV. Reducing the dimension of tensor products: power transformations

As ℓ increases, the number of repeated monomials in $\bar{x}$ becomes dominant and makes the whole machinery developed above inefficient even for problems of relatively low size. To reduce dimensions and eliminate the redundancies in the state vector, it is possible to make use of the so-called power transformations described in [6], [24]. In general, following [6], it is known that the number of linear independent homogeneous polynomials of degree ℓ in n variables is given by $r = \binom{n+\ell-1}{\ell} = \frac{(n+\ell-1)!}{n!\,\ell!}$. Call $x^{[\ell]}$ the r-dimensional vector of the lexicographically ordered (and suitably normalized) monomials: $\sqrt{\binom{\ell}{\ell_1}\binom{\ell-\ell_1}{\ell_2}\ldots\binom{\ell-\ell_1-\ldots-\ell_{n-1}}{\ell_n}}x_1^{\ell_1}x_2^{\ell_2}\ldots x_n^{\ell_n}$ such that $\sum_{i=1}^n \ell_i = \ell$, $\ell_i \geqslant 0$. Then $x^{[\ell]}$ contains the same information of $\bar{x}$ but without repeated entries. Call $\Lambda \in M_{r\times n^\ell}$ the matrix mapping between $x^{[\ell]}$ and $\bar{x}$. Its elements are:

$$(\Lambda)_{jk} = \begin{cases} \dfrac{1}{\sqrt{\binom{\ell}{\ell_1}\binom{\ell-\ell_1}{\ell_2}\ldots\binom{\ell-\ell_1-\ldots-\ell_{n-1}}{\ell_n}}} & \text{if } \begin{cases} c(k_i = 1) = \ell_1 \\ \vdots \\ c(k_i = n) = \ell_n \end{cases} \\ 0 & \text{otherwise} \end{cases}$$

where $c =$cardinality. For Λ we have that

$$\Lambda^\dagger = \Lambda^T \tag{30}$$

and therefore that $\Lambda\Lambda^T = I_r$ and $\bar{x} = \Lambda^T x^{[\ell]}$. The normalization chosen implies that the state vector $x^{[\ell]} = \Lambda\bar{x}$ is such that $\|x^{[\ell]}\| = \|\bar{x}\| = \|x\|^\ell$. Using Λ, (2) becomes $\dot{x}^{[\ell]} = A_{[\ell]}x^{[\ell]}$, with $A_{[\ell]} \in M_{r\times r}$ the "extended matrix" given by $A_{[\ell]} = \Lambda \bar{A} \Lambda^T$. The system with inputs (4) becomes:

$$\begin{aligned}\dot{x}^{[\ell]} =& A_{[\ell]} x^{[\ell]} + \Lambda \left(B \otimes I \ldots \otimes I \, u \otimes x \otimes \ldots \otimes x \right. \\ & \left. + \ldots + I \otimes \ldots \otimes I \otimes B \, x \otimes \ldots \otimes x \otimes u \right)\end{aligned} \tag{31}$$

where the inputs terms are in general different from each other. With the feedback $u = -Kx$, the closed loop system (5) becomes:

$$\dot{x}^{[\ell]} = A_{[\ell]} x^{[\ell]} + \Lambda \overline{BK} \Lambda^T x^{[\ell]} = (A - BK)_{[\ell]} \, x^{[\ell]}. \tag{32}$$

It is known, see [9], that any polynomial which is a sum-of-squares admits a "quadratic" representation via Gram matrices. For homogeneous polynomials, a 2ℓ-HP form $V(\bar{x}) = \bar{x}^T P \bar{x}$ transforms into $V(x^{[\ell]}) = x^{[\ell]^T} \Pi x^{[\ell]}$ with $\Pi \in M_r$, $\Pi = \Lambda P \Lambda^T$. For positive definite forms we have:

- $P > 0 \Longrightarrow \Pi > 0$
- $\Pi > 0 \Longrightarrow P \geqslant 0$.

The first expression follows from Λ being full rank, the second from $r < n^\ell$. Hence, sum-of-squares in $\bar{x}$ are mapped to sum-of-squares in $x^{[\ell]}$. Similarly, from (6), $\dot{V}(x^{[\ell]}) = x^{[\ell]^T} \left(A_{[\ell]}^T \Pi + \Pi A_{[\ell]} \right) x^{[\ell]}$ and

- $\bar{A}^T P + P \bar{A} < 0 \Longrightarrow A_{[\ell]}^T \Pi + \Pi A_{[\ell]} < 0$
- $A_{[\ell]}^T \Pi + \Pi A_{[\ell]} < 0 \Longrightarrow \bar{A}^T P + P \bar{A} \leqslant 0$.

It is therefore possible to re-express some of the conditions of the previous Section in terms of the power transformation basis. For example, from (30), $B^T \otimes I \otimes \ldots \otimes I \bar{x} = \left(B^T \otimes I \otimes \ldots \otimes I \right) \Lambda^T x^{[\ell]}$, hence (10) can be written as

$$\begin{aligned} B^T x = 0 &\Longleftrightarrow \left(\Lambda (B \otimes I \otimes \ldots \otimes I) \right)^T x^{[\ell]} = 0 \\ &\;\;\vdots \\ &\Longleftrightarrow \left(\Lambda (I \otimes \ldots \otimes I \otimes B) \right)^T x^{[\ell]} = 0. \end{aligned} \tag{33}$$

Concerning the LMIs, if $\Omega = \Pi^{-1}$, the expression (11) is replaced by

$$(A - BK)_{[\ell]} \, \Omega + \Omega \, (A - BK)_{[\ell]}^T < 0. \tag{34}$$

The sufficient condition (13) can be replaced by

$$\left(B^{\perp, [\ell} \right)^T \left(A_{[\ell]} \Omega + \Omega A_{[\ell]}^T \right) B^{\perp, [\ell} < 0 \tag{35}$$

where $B^{\perp, [\ell} = \Lambda \left(B^\perp \otimes \ldots \otimes B^\perp \right)$. Notice, however, that (35) is *not* equivalent to (13), because the power transformation reduces the "dimensional gap" between (34) and (35) yielding a necessary condition weaker than (13).

The formulation (32) does not simplify the feedback synthesis problem with respect to (5), as the matrix describing the gain K in the power transformed basis still has a structured form, implying a bilinear matrix inequality will still appear.

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