Following a Path of Varying Curvature as an Output Regulation Problem

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***Abstract*—Given a path of nonconstant curvature, local asymptotic stability can be proven for the general n trailer whenever the curvature can be considered as the output of an exogenous dynamical system. The controllers that provide convergence to zero of the tracking error chosen for the path-following problem are composed of a prefeedback that input–output linearizes the system, plus a linear controller.**

***Index Terms*—Output regulation, path following, wheeled vehicles.**

I. Introduction

In the several studies dealing with path following for wheeled vehicles (see, for example, [12]–[14], or the book chapters [3], [4], and [9]), convergence is usually proven for paths of constant curvature. In fact, that represents the only case in which the decoupling between lateral and longitudinal dynamics is exact for the original system and a constant steady state for the lateral dynamics exists. After fixing the longitudinal input to a nonnull constant, if the tracking error used is a scalar, the system to analyze is basically a single-input–single-output (SISO) system with drift from the steering input to the tracking error. When the curvature of a path to follow can be modeled as the output of a neutrally stable dynamical system, then the path following problem can be formulated as an output regulation problem in the nonlinear setting proposed by [8]. In fact, the curvature can be considered as a known exogenous disturbance and the output of the system, corresponding to the tracking error of the path following criterion, can be rendered independent from it by input–output linearizing the system with a static change of input. With the error independent from the curvature, if the relative degree of the system is well defined, the output zeroing manifold is the only invariant manifold that solves the regulation problem. This is equivalent to saying that local asymptotic stability to the nonconstant steady state is achieved by and only by the controllers composed of a prefeedback that input–output linearizes the system plus a linear part that can be chosen in an optimal (linear) fashion. If we choose as tracking criterion the one proposed in [2] based on the so-called off-tracking distance, whose peculiarity is that it keeps the whole vehicle (and not a single guidepoint on the vehicle) at a reduced distance from the path, then the relative degree between the steering velocity and the corresponding tracking error is equal to two, whereas for the criteria normally used, it is either higher or it is not well defined at all because of the kingpin hitches. For the same reason, properties like input-state linearization or differential flatness do not hold [11].

It must be noticed that the whole analysis is *local* and that, due to the singularities of the Frenet frame representation used here, there is no way to formally prove a well-behaved transient even for admissible initial conditions that are close to the limits of the region of attraction.

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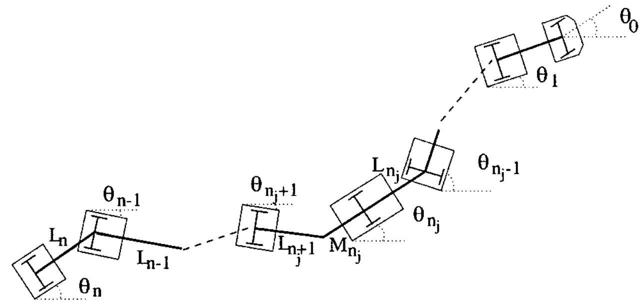

Fig. 1. General n-trailer system.

II. Kinematic Model for the General n-Trailer and Frenet Frames

Suppose we have a general n-trailer system with m $(m \leq n)$ of the trailers hooked at a distance M_i from the preceding axle; see Fig. 1. Assume that each body is composed of one single axle. The nonholonomic constraints on the points P_i (below called *nonholonomic points*) originate from the assumption of rolling without slipping of the wheels. If we call $n_1, \ldots, n_m$, $n_j < n_{j+1}$, $n_m < n$ the indices of the axles having nonnull off-hitching $(M_{n_j} \neq 0)$ we can group together the axles between two consecutive kingpin hitchings: $\{0, 1, \ldots, n_1\}, \ldots, \{n_{j-1}+1, n_{j-1}+2, \ldots, n_j-1, n_j\}, \ldots, \{n_m+1, n_m+2, \ldots, n-1, n\}$. We do not consider the case of two consecutive axles having off-hitching. Call θ_i the orientation angle of the ith axle, v_i its translational velocity, L_i the distance between the ith axle and the hitching point of the same trailer, and $\beta_1 \triangleq \theta_0 - \theta_1$ the steering angle. The n-trailer system has two inputs, corresponding to translational and steering actions of the car pulling the trailers. At the kinematic level, we can consider these two inputs to be the steering speed $\omega \triangleq \dot{\beta}_1$ and the translational speed v_n of the last trailer.

Under the assumption that the path is sufficiently smooth and that the curvature has an upper bound, a particularly useful local frame to describe the lateral dynamics of the path following problem decoupled from the longitudinal one is the so-called Frenet frame i.e., a frame moving on the path having origin on the orthogonal projection of the point of interest. In [2], the tracking criterion introduced consists in considering $n+1$ frames simultaneously, one for each nonholonomic point. Each of the curvilinear frames (see Fig. 2) is represented by two coordinates $(s_{\gamma_i}, \theta_{\gamma_i})$ where s_{γ_i} is the line integral along the path to follow, up to the actual projection of the point P_i on the path itself and θ_{γ_i} is the orientation of the frame with respect to the inertial frame. In the Frenet frame, the point P_i is represented by the signed distance z_i between the point itself and its orthogonal projection and by the relative orientation angle $\tilde{\theta}_i$. The decoupling property of the Frenet frame has already been used by several authors for the path following problem (see [10] and [12]). We also use it but substituting the tracking criterion normally used [13]

$$z_n \to 0 \tag{1}$$

or an equivalent one based on another of the distances z_i, with the *sum* of the signed distances

$$\sum_{i=0}^{n} z_i \to 0. \tag{2}$$

It can be noticed that for a nonzero curvature neither the θ_i nor the θ_{γ_i} tend to a steady state in the path-following problem, but their difference $\tilde{\theta}_i \triangleq \theta_i - \theta_{\gamma_i}$, $i \in \{0, 1, \ldots, n\}$ can have an equilibrium value if

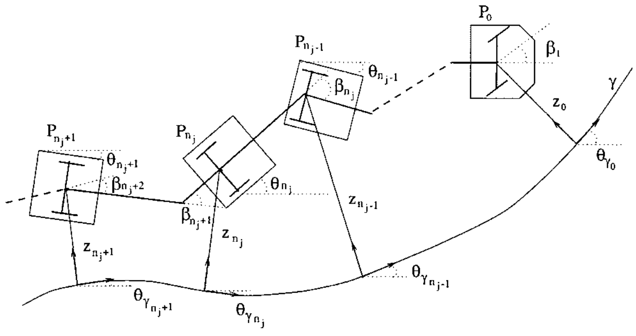

Fig. 2. Frenet frames associated with the nonholonomic points P_i.

the curvature $\kappa_\gamma =$ const. The same observation is valid also for the angles $\beta_i \triangleq \theta_{i-1} - \theta_i$, $i \in \{1, \dots, n\}$. Therefore, it is convenient to write the dynamic equations of θ_i and θ_{γ_i} in terms of $\tilde\theta_i$ and β_i. We can group together all four equations relative to each point P_i. When there is off-axle hitching the equations for the node P_{n_j+1} (first node of each steering train, except for the driving cart) are shown in (3) at the bottom of the page, $j \in \{0, 1, \dots, m\}$, $n_0 = 0$, and $n_1 > 1$. For the other nonholonomic points, the corresponding M_{n_j+i} are zero. Therefore, the formulas simplify to

$$\begin{bmatrix}\dot s_{\gamma_{n_j+i}}\\ \dot z_{n_j+i}\\ \dot{\tilde\theta}_{n_j+i}\\ \dot\beta_{n_j+i}\end{bmatrix} = v_{n_j+i}\begin{bmatrix} \dfrac{\cos\tilde\theta_{n_j+i}}{1-\kappa_\gamma\left(s_{\gamma_{n_j+i}}\right)z_{n_j+i}} \\ \sin\tilde\theta_{n_j+i} \\ \dfrac{\tan\beta_{n_j+i}}{L_{n_j+i}} - \dfrac{\cos\tilde\theta_{n_j+i}\kappa_\gamma\left(s_{\gamma_{n_j+i}}\right)}{1-\kappa_\gamma\left(s_{\gamma_{n_j+i}}\right)z_{n_j+i}} \\ \dfrac{\tan\beta_{n_j+i-1}}{L_{n_j+i-1}\cos\beta_{n_j+i}} - \dfrac{\tan\beta_{n_j+i}}{L_{n_j+i}}\end{bmatrix} \tag{4}$$

$j \in \{0, 1, \dots, m\}$, $i \in \{2, 3 \dots, n_{j+1} - n_j\}$, $n_{m+1} = n$, where (5), shown at the bottom of the page, holds; $j \in \{0, 1, \dots, m\}$, $i \in \{1, 2 \dots, n_{j+1} - n_j\}$, and $v_0 = v_1/\cos\beta_1$. Considering v_n to be a given (non-null) open-loop function, for example a constant, we obtain a system with a drift component. The domain of definition $\mathcal{D}$ and the singularity locus of the general n trailer are discussed in [1]. Calling $\boldsymbol{p}$ the state vector

$$\boldsymbol{p} = [\, s_{\gamma_n}\; z_n\; \tilde\theta_n\; \beta_n \quad \dots \quad s_{\gamma_1}\; z_1\; \tilde\theta_1\; \beta_1 \quad s_{\gamma_0}\; z_0\; \tilde\theta_0 \,]^T$$

the dynamic equations of the system are

$$\dot{\boldsymbol{p}} = \mathcal{F}(\boldsymbol{p}) + \mathcal{G}\omega, \qquad \boldsymbol{p} \in \mathcal{D} \tag{6}$$

where the input is entering only in the differential equations of $\tilde\theta_0$ and β_1. In order to consider simultaneously the error distances of all the nonholonomic points P_i from the path, we take as output the sum of the $n+1$ signed distances z_i

$$y = [\,0100 \quad 0100 \quad \dots \quad 0100 \quad 010]\boldsymbol{p} \triangleq \mathcal{H}\boldsymbol{p}. \tag{7}$$

Considering multiple frames on the same rigid body leads to a redundant description of the system. To recover the original dynamic equations, a number of constraints must be added. However, for our stabilization purposes they can simply be neglected and we can work with the overparameterized system (6) and (7); see again [2] for a complete formulation.

III. Input–Output Feedback Linearization

In what follows, we will assume that $M_0 = 0$, i.e., that there is no off-axle connection on the driving unit. In fact, if $M_0 \neq 0$ the general n-trailer does not have a well-defined relative degree. For this and the other concepts used in the remaining of the paper (Lie derivative, input-state and input–output feedback linearization, zero dynamics, etc.), we remand the reader to any standard text on nonlinear control systems, such as [7].

The following proposition can be proven by direct calculation.

Proposition 1: The n-trailer system (6) and (7) with the tracking criterion (2) has relative degree two.

The low relative degree suggests that input–output feedback linearization is easily attained for our system: in fact it is enough to differentiate the output (7) twice and cancel the corresponding dynamics by means of a change of input. From

$$y = \sum_{i=0}^{n} z_i$$

we get the equation shown at the bottom of the next page. In order to have a well-defined relative degree, we have already assumed that the driving unit has no off-axle connection, i.e., $M_0 = 0$. For sake of simplicity, we require here also that $M_1 = 0$. The case with $M_1 \neq 0$ does not differ except for the more involved formulation of the domain

$$\begin{bmatrix}\dot s_{\gamma_{n_j+1}}\\ \dot z_{n_j+1}\\ \dot{\tilde\theta}_{n_j+1}\\ \dot\beta_{n_j+1}\end{bmatrix} = v_{n_j+1}\begin{bmatrix} \dfrac{\cos\tilde\theta_{n_j+1}}{1-\kappa_\gamma\left(s_{\gamma_{n_j+1}}\right)z_{n_j+1}} \\ \sin\tilde\theta_{n_j+1} \\ \dfrac{\tan\beta_{n_j+1} - \frac{M_{n_j}}{L_{n_j}}\tan\beta_{n_j}}{L_{n_j+1}\left(1+\frac{M_{n_j}}{L_{n_j}}\tan\beta_{n_j}\tan\beta_{n_j+1}\right)} - \dfrac{\cos\tilde\theta_{n_j+1}\kappa_\gamma\left(s_{\gamma_{n_j+1}}\right)}{1-\kappa_\gamma\left(s_{\gamma_{n_j+1}}\right)z_{n_j+1}} \\ \dfrac{\tan\beta_{n_j}}{L_{n_j}\cos\beta_{n_j+1}} - \dfrac{\tan\beta_{n_j+1}}{L_{n_j+1}} + \dfrac{M_{n_j}\tan\beta_{n_j}}{L_{n_j+1}L_{n_j}}\end{bmatrix} \tag{3}$$

$$v_{n_j+i} = \frac{v_n}{\prod_{k=j+1}^{m}\left(1+\frac{M_{n_k}}{L_{n_k}}\tan\beta_{n_k}\tan\beta_{n_k+1}\right)\prod_{k=n_j+i+1}^{n}(\cos\beta_k)} \tag{5}$$

of definition and will be treated in the example of Section V. In fact, when differentiating the above expression a second time we need to isolate the terms in $\tilde{\theta}_0$ and β_1 whose derivatives introduce the input ω. With $M_0 = M_1 = 0$ they appear only in the term $v_0 \sin\tilde{\theta}_0$. Calling $\mathfrak{p}$ the state obtained from $\boldsymbol{p}$ excluding β_1 and $\tilde{\theta}_0$, we have

$$\begin{aligned}\ddot{y} =& L_{\mathcal{F}}^2\mathcal{H}\boldsymbol{p} + L_{\mathcal{G}}L_{\mathcal{F}}\mathcal{H}\boldsymbol{p}\omega = \frac{\partial\dot{y}}{\partial\mathfrak{p}}\mathfrak{p} + \frac{\partial\dot{\mathfrak{y}}}{\partial\beta_1}\dot{\beta}_1 + \frac{\partial\dot{\mathfrak{y}}}{\partial\tilde{\theta}_0}\dot{\tilde{\theta}}_0\\ =& \frac{\partial\dot{y}}{\partial\mathfrak{p}}\mathfrak{p} + \mathfrak{v}_0^2\cos\tilde{\theta}_0\left(\frac{\sin\beta_1}{\mathfrak{L}_1} - \frac{\cos\tilde{\theta}_0\kappa_\gamma(\mathfrak{s}_{\gamma_0})}{1-\kappa_\gamma(\mathfrak{s}_{\gamma_0})\mathfrak{z}_0}\right)\\ &+ v_0\left(\sin\tilde{\theta}_0\tan\beta_1 + \cos\tilde{\theta}_0\right)\omega.\end{aligned}$$

The term $\left(\sin\tilde{\theta}_0\tan\beta_1 + \cos\tilde{\theta}_0\right)$ vanishes $\iff \tan(\tilde{\theta}_0 - \beta_1) = \pm\infty \iff \tilde{\theta}_0 - \beta_1 = (\pi/2) \bmod \pi$. Since $\tilde{\theta}_0 - \beta_1 = \theta_1 - \theta_{\gamma_0}$ the singularities are function of how much the path is "bending" between the projections on the path of P_0 and P_1. Therefore, for $\boldsymbol{p}$ in $\mathcal{D} \cap \left\{(\tilde{\theta}_0, \beta_1) \text{ s.t. } \tilde{\theta}_0 - \beta_1 \in]-\pi/2, \pi/2[\right\}$, the input transformation

$$\begin{aligned}\omega =& \frac{-L_{\mathcal{F}}^2\mathcal{H}\boldsymbol{p} + u}{L_{\mathcal{G}}L_{\mathcal{F}}\mathcal{H}\boldsymbol{p}}\\ =& \frac{-\left(\frac{\partial\dot{y}}{\partial\mathfrak{p}}\mathfrak{p} + \mathfrak{v}_0^2\cos\tilde{\theta}_0\left(\frac{\sin\beta_1}{\mathfrak{L}_1} - \frac{\cos\tilde{\theta}_0\kappa_\gamma(\mathfrak{s}_{\gamma_0})}{1-\kappa_\gamma(\mathfrak{s}_{\gamma_0})\mathfrak{z}_0}\right)\right) + u}{v_0\left(\sin\tilde{\theta}_0\tan\beta_1 + \cos\tilde{\theta}_0\right)}\end{aligned} \tag{8}$$

is a diffeomorphism that reduces the input–output dynamics to the chain of integrators

$$\ddot{y} = u \tag{9}$$

that can be stabilized using linear control theory provided that the system is minimum phase. The zero dynamics is obtained confining the dynamics of the system to the so-called *output-zeroing manifold*

$$Z^* = \{\boldsymbol{p} \in \mathcal{D} \text{ s.t. } y = \dot{y} = \ddot{y} = 0\}.$$

In practice, it is obtained by adding to the original system (6) the conditions $y = 0$, $\dot{y} = 0$ and the input

$$\omega = \frac{-L_{\mathcal{F}}^2\mathcal{H}\boldsymbol{p}}{L_{\mathcal{G}}L_{\mathcal{F}}\mathcal{H}\boldsymbol{p}} \tag{10}$$

and it represents the part of the system equations which is no longer connected to the output after the change of input.

Proposition 2: The zero dynamics of the n-trailer system (6) and (7) is locally asymptotically stable along paths of constant curvature.

Proof: If in a generic n-trailer system we take as tracking criterion $z_0 \to 0$, then for forward motions the stability of the zero dynamics on the output zeroing manifold $\hat{Z}^*$ corresponding to the output $\hat{y} = \hat{\mathcal{H}}\boldsymbol{p} = z_0$ is immediate to understand as it corresponds to have the nonholonomic point P_0 exactly on the path for all times. Differentiating $\hat{y}$

$$\begin{aligned}\dot{\hat{y}} =& v_0\sin\tilde{\theta}_0\\ \ddot{\hat{y}} =& L_{\mathcal{F}}^2\hat{\mathcal{H}}\boldsymbol{p} + L_{\mathcal{G}}L_{\mathcal{F}}\hat{\mathcal{H}}\boldsymbol{p}\omega\end{aligned}$$

with $L_{\mathcal{G}}L_{\mathcal{F}}\hat{\mathcal{H}}\boldsymbol{p} = L_{\mathcal{G}}L_{\mathcal{F}}\mathcal{H}\boldsymbol{p}$ when $M_1 = 0$. The zero dynamics of such a tracking error, given by $z_0 = \tilde{\theta}_0 = 0$ plus a feedback similar to (10), is trivially asymptotically stable with exponential rate of convergence for forward motion and for any admissible initial condition on $\hat{Z}^*$. The state equations on $\hat{Z}^*$ (to which the constraints $\hat{y} = \dot{\hat{y}} = 0$ must be added) look like

$$\dot{\boldsymbol{p}} = \mathcal{F}(\boldsymbol{p}) - \mathcal{G}\frac{L_{\mathcal{F}}^2\hat{\mathcal{H}}\boldsymbol{p}}{L_{\mathcal{G}}L_{\mathcal{F}}\hat{\mathcal{H}}\boldsymbol{p}}. \tag{11}$$

It corresponds to P_0 (where the steering input is applied) on the path and wrong initial conditions on the trailers and it asymptotically decays to an equilibrium point which is unique for a path of given constant curvature. Returning to the tracking error (2), the equilibrium point for (6) and (7) around which to check convergence of the zero dynamics in Z^*, call it $\boldsymbol{p}_e$, is computed in detail in [2] and it still corresponds to a circular concentric trajectory. Writing the output (7) as $y = \hat{y} + \check{y} = \hat{\mathcal{H}}\boldsymbol{p} + \check{\mathcal{H}}\boldsymbol{p}$, we have $L_{\mathcal{G}}L_{\mathcal{F}}\check{\mathcal{H}}\boldsymbol{p} = 0$, i.e., the relative degree with respect to $\check{y}$ is three or higher, thus the state-space equations on Z^* will be given by

$$\dot{\boldsymbol{p}} = \mathcal{F}(\boldsymbol{p}) - \mathcal{G}\frac{L_{\mathcal{F}}^2\hat{\mathcal{H}}\boldsymbol{p}}{L_{\mathcal{G}}L_{\mathcal{F}}\hat{\mathcal{H}}\boldsymbol{p}} - \mathcal{G}\frac{L_{\mathcal{F}}^2\check{\mathcal{H}}\boldsymbol{p}}{L_{\mathcal{G}}L_{\mathcal{F}}\hat{\mathcal{H}}\boldsymbol{p}}. \tag{12}$$

By comparison with (11), local asymptotic stability of (12) around $\boldsymbol{p}_e$ follows from exponentially stability on the large of (11). ■

The chain of integrators (9) can now be stabilized for example using linear quadratic theory. Any output feedback of the form

$$u = k_1 y + k_2\dot{y} \tag{13}$$

with $k_1 < 0, k_2 < 0$ is a locally asymptotically stabilizer for the whole system.

IV. Following a Path of Varying Curvature as an Output Regulation Problem

In what follows, we will try to asymptotically stabilize the system to paths whose curvature is varying in a given class of functions. We will treat the problem as an *output regulation problem* in which the error $y(\cdot)$ has to asymptotically reject the variation of curvature $\kappa_\gamma(s_\gamma)$ regarded as a persistent input generated by a dynamical system. In the classical context of linear time-invariant, finite-dimensional systems, this geometric control problem was first solved by Davison [5] and Francis and Wonham [6] based on the assumption that the external command can be modeled as the output of an autonomous system called the *exosystem*. The solution was then extended to the nonlinear case by Isidori and Byrnes [8]. The presence of a known "disturbance" acting as a persistent input implies that the steady state of the system is varying depending only on the exogenous input and not on the initial conditions of the system (that have to be in an appropriate neighborhood of the origin). In our case, the exogenous input of the system is the curvature function $\kappa_\gamma(\cdot)$ of the path. To be consistent with our control problem, the curvature has to be upper bounded; in fact, too high a curvature implies that a steady state for the tracking criterion (2) does not exist.

The properties of persistence in time and of boundedness of the exogenous input are compactly described by the notion of *neutral stability* of the exogenous system. A system is said neutrally stable if it

$$\begin{aligned}\dot{y} =& L_{\mathcal{F}}\mathcal{H}\boldsymbol{p} = \sum_{i=0}^{n} v_i\sin\tilde{\theta}_i\\ =& \sum_{j=0}^{m}\sum_{i=1}^{n_{j+1}-n_j}\frac{v_n\sin\tilde{\theta}_{n_j+i}}{\prod_{k=j+1}^{m}\left(1+\frac{M_{n_k}}{L_{n_k}}\tan\beta_{n_k}\tan\beta_{n_k+1}\right)\prod_{k=n_j+i+1}^{n}(\cos\beta_k)} + v_0\sin\tilde{\theta}_0.\end{aligned}$$

is both Lyapunov stable and Poisson stable. A necessary condition for a system to be neutrally stable is that its first-order approximation has all the eigenvalues on the imaginary axis. In our case, the exogenous system has to represent how the curvature is evolving along the path γ. Looking at (6), we can see that at every time instant the curvature has $n+1$ "entries" in the equations, corresponding to the values of curvature in different positions along the path. For the nonholonomic point P_i, if the curvature function is given in terms of the curvilinear abscissa $\kappa_{\gamma_i} = \kappa_\gamma(s_{\gamma_i})$ then we can think of it as generated by a dynamical system

$$\kappa'_{\gamma_i} = \frac{d\kappa_{\gamma_i}}{ds_{\gamma_i}} = \gamma(\kappa_{\gamma_i}), \qquad i = 0, 1, \ldots n \tag{14}$$

where the independent variable is the curvilinear abscissa s_{γ_i}. In order to couple this exogenous system with the remaining part of the equations, we have to rescale it as a function of time, expressing s_{γ_i} as $s_{\gamma_i}(t)$, i.e., substituting the space derivatives of (14) with the corresponding time derivatives

$$\dot{\kappa}_{\gamma_i} = \frac{d\kappa_{\gamma_i}}{dt} = \frac{d\kappa_{\gamma_i}}{ds_{\gamma_i}} \frac{ds_{\gamma_i}}{dt} = v_{\gamma_i}\gamma(\kappa_{\gamma_i}) = \dot{s}_{\gamma_i}\gamma(\kappa_{\gamma_i}). \tag{15}$$

The exogenous equation is the same for all the nonholonomic points (since the reference path is the same) but the initial values of κ_{γ_i} are different since they express the value of the curvature at the initial curvilinear abscissa $s_{\gamma_i}(0)$. The presence of the term $\dot{s}_{\gamma_i}$ does not spoil the "exogenousness" of (15): it is in fact possible to rescale the whole system (6) as a function of the curvilinear abscissa and of the "spatial" integral constraints mentioned in Section II and reported in detail in [2], yielding, in principle, a completely time-independent system in which the terms $\dot{s}_{\gamma_i}$ obviously disappear. Provided we can prove well posedness and asymptotic stability of the problem in the time-dependent scale, then in the formulation (15) the $\dot{s}_{\gamma_i}$ represent terms which are monotone, bounded and continuous (for paths of continuous curvature and for P_i near the path) since they represent the projections on the path of the translational velocities v_i of the nonholonomic points P_i:$0 < \dot{s}_{\gamma_i} \leq v_i$. Therefore the neutral stability of (14) implies the neutral stability of (15) and vice-versa. In fact, the eigenvalues of the first order approximation are on the imaginary axis in both cases. For all times t we have $s_{\gamma_n}(t) < s_{\gamma_{n-1}}(t) < \cdots < s_{\gamma_0}(t)$, but the delay between $s_{\gamma_i}(t)$ and $s_{\gamma_{i-1}}(t)$ is variable according to the curvature of the path in the interval $s_{\gamma_{i-1}}(t) - s_{\gamma_i}(t)$ and to the position and orientation of the vehicle with respect to the path. Calling $\kappa_\gamma = [\kappa_{\gamma_n} \quad \cdots \quad \kappa_{\gamma_1} \quad \kappa_{\gamma_0}]^T$ and $s_\gamma = [s_{\gamma_n} \quad \cdots \quad s_{\gamma_1} \quad s_{\gamma_0}]^T$, the complete system is then

$$\dot{\boldsymbol{p}} = \mathcal{F}(\boldsymbol{p}, \kappa_\gamma) + \mathcal{G}(\boldsymbol{p})\omega \tag{16}$$

$$\dot{\kappa}_\gamma = \dot{s}_\gamma^T \Gamma(\kappa_\gamma) \tag{17}$$

$$y = \mathcal{H}\boldsymbol{p} \tag{18}$$

where $\Gamma(\kappa_\gamma)$ has the diagonal structure

$$\begin{bmatrix} \gamma(\kappa_{\gamma_n}) & & 0 \\ & \ddots & \\ 0 & & \gamma(\kappa_{\gamma_0}) \end{bmatrix}.$$

The right formulation for our case is called *full information output regulation problem*, in which the whole state of the system is measurable. Here, we follow the definition given in [7].

Given the nonlinear system (16) and the neutrally stable exogenous system (17), the output regulation problem is said to be solvable if there exists a map $\alpha(\boldsymbol{p}, \kappa_\gamma)$ such that

P1) equilibrium $\boldsymbol{p} = 0$ of

$$\dot{\boldsymbol{p}} = \mathcal{F}(\boldsymbol{p}, 0) + \mathcal{G}(\boldsymbol{p})\alpha(\boldsymbol{p}, 0)$$

is asymptotically stable in the first order approximation;

P2) there exist a neighborhood $V \subset \Pi \times K_\Gamma^0$ of $(0,0)$ such that for each initial condition $(\boldsymbol{p}(0), \kappa_\gamma(0)) \in V$, the solution of (16) satisfies

$$\lim_{t\to\infty} \mathcal{H}\boldsymbol{p}(t) = 0.$$

The statement P1) is motivated by the center manifold theory. In fact, given (16) and (17), we know that the eigenvalues of the exogenous system are on the imaginary axis and cannot be moved. Therefore, the problem is solvable only if all the other eigenvalues of the system can be moved to the open left half of the complex plane by means of a state feedback on the endogenous input ω. If such a feedback can be found for $\kappa_\gamma = 0$, then the center manifold theory assures the existence of an invariant manifold in a neighborhood of the origin whose graph is the solution of an associated partial differential equation. This is formulated in the following theorem.

Theorem 1 [7]: Given the neutrally stable system (17) and assuming the existence of an endogenous feedback law $\omega = \alpha(\boldsymbol{p}, 0), \alpha(0,0) = 0$ such that the equilibrium $\boldsymbol{p} = 0$ of

$$\dot{\boldsymbol{p}} = \mathcal{F}(\boldsymbol{p}, 0) + \mathcal{G}(\boldsymbol{p})\alpha(\boldsymbol{p}, 0)$$

is asymptotically stable in the first-order approximation, then there exist mappings $\boldsymbol{p} = \pi(\kappa_\gamma)$ and $\omega = \alpha(\pi(\kappa_\gamma), \kappa_\gamma)$ defined in a neighborhood $K_\Gamma^\circ \subset K_\Gamma$ of the origin with $\pi(0) = 0$ and $\alpha(0,0) = 0$, which satisfy

$$\frac{\partial \pi}{\partial \kappa_\gamma}\Gamma(\kappa_\gamma) = \mathcal{F}(\pi(\kappa_\gamma), \kappa_\gamma) + \mathcal{G}(\pi(\kappa_\gamma))\alpha(\pi(\kappa_\gamma), \kappa_\gamma)$$

$\forall\, \kappa_\gamma \in K_\Gamma^\circ$.

The theorem assures also the existence of a well-defined steady-state response for every exogenous input in K_Γ°.

Consider the Jacobian of $\mathcal{F}$ at the origin

$$\mathcal{F}_{e_0} = \left.\frac{\partial \mathcal{F}(\boldsymbol{p}, \kappa_\gamma)}{\partial \boldsymbol{p}}\right|_{(0,0)}.$$

From linear control theory, it is deduced that the stabilizability of the pair $(\mathcal{F}_{e_0}, \mathcal{G})$ is also a necessary condition for the solution of P1.

The previous condition can be used to adapt the necessary and sufficient condition for the solution of the full information output regulation problem provided in [7] to our case.

Theorem 2: Given (16)–(18) with (17) neutrally stable, the full information output regulation problem is solvable if and only if $(\mathcal{F}_{e_0}, \mathcal{G})$ is stabilizable and there exist mappings $\boldsymbol{p} = \pi(\kappa_\gamma)$ and $\omega = c(\kappa_\gamma)$ with $\pi(0) = 0$ and $c(0) = 0$, both defined in a neighborhood $K_\Gamma^\circ \subset K_\Gamma$ satisfying the conditions

$$\frac{\partial \pi}{\partial \kappa_\gamma}\Gamma(\kappa_\gamma) = \mathcal{F}(\pi(\kappa_\gamma), \kappa_\gamma)) + \mathcal{G}(\pi(\kappa_\gamma))c(\kappa_\gamma) \tag{19}$$

$$0 = \mathcal{H}\pi(\kappa_\gamma) \tag{20}$$

for all $\kappa_\gamma \in K_\Gamma^\circ$.

Conditions (19) and (20) express the fact that the mapping $\boldsymbol{p} = \pi(\kappa_\gamma)$ which is rendered locally invariant by the feedback law $\omega = c(\kappa_\gamma)$ has to be an output zeroing manifold of the composite system. In our case, due to the independence of the output (18) from the curvature κ_γ, the output zeroing property is not related to the exogenous system but only to the exact input–output feedback linearization. This is formalized in the following theorem.

Theorem 3: For (16)–(18) with (17) neutrally stable, the full information output regulation problem is solvable. All the controllers that solve the problem are composed of a prefeedback (8) that input–output exactly linearizes the system and of a stabilizing feedback for the resulting chain of integrators.

Proof: Apply the input–output linearizing controller (8) to (16)–(18). The output function $\mathcal{H}\boldsymbol{p}$ and its first Lie derivative $L_\mathcal{F}\mathcal{H}\boldsymbol{p}$

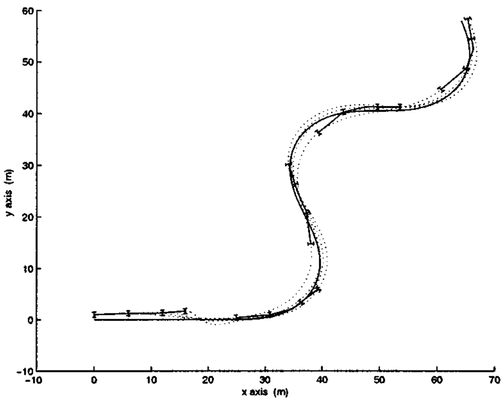

Fig. 3. Following a path of sinusoidal curvature with the linear controller proposed in [2].

constitute the first components of a change of basis $\xi = \Phi(\boldsymbol{p})$ that, together with the feedback (8), transforms the system into normal form

$$\begin{aligned} \dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= u \\ \dot{\bar{\xi}} &= \mathcal{A}(\xi, \kappa_\gamma) + \mathcal{B}(\xi, \kappa_\gamma, u) \\ y &= \xi_1. \end{aligned} \tag{21}$$

The subsystem $\bar{\xi} = [\xi_3 \dots \xi_n]^T$ which corresponds to the zero dynamics was shown to be asymptotically stable in Proposition 2. Therefore, the whole system is locally asymptotically stabilizable around $\xi = 0$ for $\kappa_\gamma = 0$ by a linear controller and, by Theorem 1 we deduce the existence of an invariant manifold characterized by the map $\xi = \pi(\kappa_\gamma)$ such that on its graph the condition (19) is satisfied. In particular, we can consider a map $\pi(\cdot)$ such that it does not touch the first two components

$$\xi_i = \pi_i(\kappa_\gamma) = \pi_i(0) = 0, \qquad i = 1, 2$$

while perturbing all the other states. A map with such characteristics fulfills also (20) when $u = 0$. Therefore, also Theorem 2 holds and the full information output regulation problem is solvable. Since the change of basis $\Phi(\cdot)$ is a local diffeomorphism and the feedback (8) is invertible, the problem is solvable also in the original basis.

By Theorem 2, a necessary condition for the solvability of the full information output regulation problem is that $\pi(\cdot)$ is an output zeroing manifold. Since the output function does not depend directly on κ_γ and the system has a locally well-defined relative degree, also Z^* is independent of κ_γ and the feedback $\omega = c(\kappa_\gamma)$ which renders each $\pi(\cdot)$ invariant in Theorem 2, is uniquely given by (10). The full controller then is obtained by adding a stabilizing loop around (8). ■

In particular, the simplest class of controllers that satisfies Theorem 3 is given by

$$\omega = \frac{-L_{\mathcal{F}}^2 \mathcal{H} \boldsymbol{p} + k_1 y + k_2 \dot{y}}{L_{\mathcal{G}} L_{\mathcal{F}} \mathcal{H} \boldsymbol{p}} \tag{22}$$

$\forall\, k_1 < 0, k_2 < 0$.

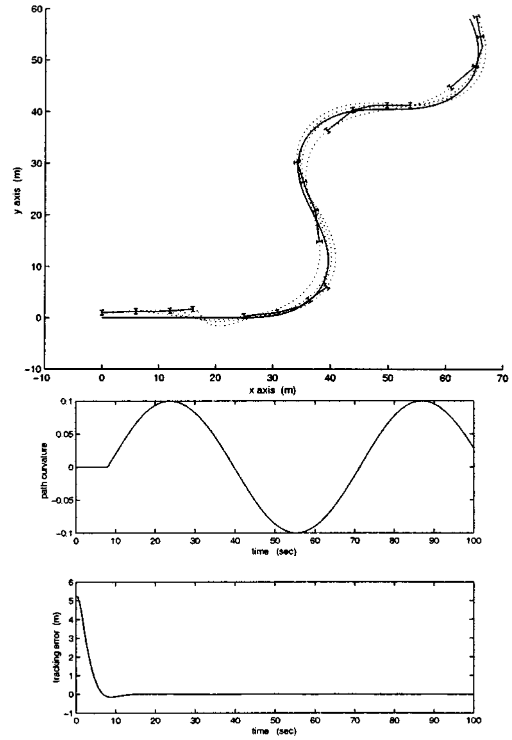

Fig. 4. Following the same sinusoidal path of Fig. 3 with the controller (22).

Such a property is characteristic not only of our system (16)–(18) but of any control-affine SISO system with relative degree for which only disturbance rejection is required, i.e., in which the exogenous system consists only of disturbances acting on the state space and not of signals to be tracked by the output. What this means is that in the case of well-defined relative degree there is no need to solve a partial differential equation to find the invariant manifold $\pi(\cdot)$, since the prefeedback (8) provides the unique solution.

V. Example

Consider a car pulling two trailers the first of which has off-axle hooking. Here, we have that $M_1 \neq 0$; therefore, the term $L_{\mathcal{G}} L_{\mathcal{F}} \mathcal{H} \boldsymbol{p}$ instead of having the expression in the denominator of (8) has the more complex one, as shown in the equation at the bottom of the page. In a neighborhood of $\boldsymbol{p} = 0$, $\cos\tilde{\theta}_0$ is the dominant term; therefore, as in (8), we can conclude that there exists a subdomain of $\mathcal{D}$ in which the denominator $L_{\mathcal{G}} L_{\mathcal{F}} \mathcal{H} \boldsymbol{p}$ is nonvanishing.

The different behaviors of the linear controller used in [2] and of the input–output linearizing controller (22) are compared for a sinusoidal path in Figs. 3 and 4. The linear controller cannot achieve any steady state even though the tracking error remains bounded. For the second controller instead, the tracking error asymptotically converges to zero. The reduced difference in the error dynamics between Figs. 3 and 4 suggests that alternative approaches to the exact linearization solution, for example considering uniform boundedness of the tracking error instead of asymptotic convergence, could be successfully applied also to simple linear controllers. On the other hand, it is worth mentioning

$$L_{\mathcal{G}} L_{\mathcal{F}} \mathcal{H} \boldsymbol{p} = v_3 \frac{\cos\beta_1 \cos\beta_2 \cos\beta_3 \left[\sin\tilde{\theta}_0 \left(\tan\beta_1 - \frac{M_1}{L_1} \tan\beta_2 \right) + \cos\tilde{\theta}_0 \left(1 + \frac{M_1}{L_1} \tan\beta_1 \tan\beta_2 \right) \right] + \sin\tilde{\theta}_1 \left(\frac{M_1}{L_1} \tan\beta_2 \right)}{\left(1 + \frac{M_1}{L_1} \tan\beta_1 \tan\beta_2 \right)^2 \cos^2\beta_1 \cos^2\beta_2 \cos^2\beta_3}.$$

that a major advantage of the feedback linearization technique is that the feedback (10) provides the open-loop control that exactly steers the system on a given path. This is normally of great help in motion planning problems.

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On the Relationship Between the Sample Path and Moment Lyapunov Exponents for Jump Linear Systems

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***Abstract*—In this note, we study the relationship between the sample and moment Lyapunov exponents for jump linear systems. Using a large deviation theorem, a modified version of Arnold's formula for connecting sample path and moment Lyapunov exponents for continuous-time linear stochastic systems is extended to discrete-time jump linear systems. Sample path stability properties of linear stochastic systems are determined by the top Lyapunov exponent and relating sample and moment Lyapunov exponents may be useful for developing computationally efficient methods for determining the almost-sure (sample path) stability of linear stochastic systems.**

***Index Terms*—Finite-state Markov chain, large deviation, linear stochastic systems, Lyapunov exponents, moment Lyapunov exponents.**

I. INTRODUCTION

Determining the stability of a linear stochastic system is an important problem. In general, the most useful stability criteria involve sample-path or almost-sure stability of the system. Necessary and sufficient conditions for sample-path stability often require a difficult computation of the top Lyapunov exponent. Although moment stability calculations, e.g., stability of the mean or the second moment, only require the stability analysis of a deterministic system, the results might not be useful in practice. In particular, for a linear stochastic system, it is well known that second-moment stabilty implies sample-path stability, but often times second moment stability criteria are too conservative to be useful in applications [11]. In this note, we investigate extending Arnold's formula relating sample and moment Lyapunov exponents for continuous-time linear stochastic systems with diffusion-type processes to discrete-time linear systems with random jump processes. The eventual goal is to use the relationship between sample and moment Lyapunov exponents to develop computationally efficient procedures for evaluating the sample-path stability of discrete-time jump linear systems.

Consider the discrete-time system

$$x(k+1) = A_k x(k) \quad x(0) = x_0 \tag{1.1}$$

where $\{A_k\}_{k\geq 0}$ is a sequence of $Gl(d, R)$-valued random variables. Here, $Gl(d, R)$ is the general linear group of dimension d over the real field, R. Fixing coordinates, a representative element of $Gl(d, R)$ is a nonsingular $d \times d$ matrix over R. A sample trajectory of (1.1) is given by the action of a random matrix product on a point $x_0 \in R^d$. Our analysis is restricted to random matrices in $Gl(d, R)$ because of the importance of *regularity* of (1.1) [3].

The asymptotic behavior of sample trajectories of system (1.1) have been studied extensively by many researchers, most notably in the context of random matrix products (see [3]). Furstenberg and Kifer [6] considered the Lyapunov exponents and the corresponding subspace filtration of the state space, and obtained an integrability condition. Arnold [1] and Arnold *et al.* [2] have been studying moment Lyapunov exponents for linear stochastic systems and discovered a formula that

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