# Coherent control of open quantum dynamical systems

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A systematic analysis of the behavior of the quantum Markovian master equation driven by coherent control fields is proposed. Its irreversible character is formalized using control-theoretic notions and the sets of states that can be reached via coherent controls are described. The analysis suggests to what extent (and how) it is possible to counteract the effect of dissipation.

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## I. INTRODUCTION

For an open quantum dynamical system (see [1–3]), the problem of coherent manipulation of the state is far less trivial than for a closed system and is the subject of recent intense research, especially but not exclusively in connection with quantum information processing; see [4–6] or the survey papers [7,8]. The (well-known) reason is that the dissipation/relaxation part of the dynamics (often called the Lindbladian) tends irreversibly to bring the system to an equilibrium point which cannot be fully compensated by means of coherent control authority, although it can be modified to some extent (see [9]). For the Markovian formulation, this is encoded in the structure of the Lindbladian. In this work, the idea of "irreversibility in spite of the control action" is made rigorous by using standard notions from classical control theory. For example, it is shown that the sets of states reachable by means of coherent controls are almost always open and dense in the space of density operators. This, in control terms the so-called *accessibility* property, is not sufficient to guarantee reversibility. The unavoidable irreversibility can be formulated as the lack of the so-called *small-time controllability*, i.e., as the impossibility of reaching arbitrary neighborhoods of a given state by means of coherent controls alone. Small-time controllability itself is only a sufficient condition for *controllability* (meaning the common intuitive notion of arbitrary manipulability of the state). In fact, we will see that depending on the structure of the Lindbladian it may happen that some target states may be reachable if we allow a long enough time to elapse. Nevertheless, it is not possible to steer *any* initial state to *any* target density in finite time. This may be achievable (again depending on the value of the Lindbladian) only as time goes to infinity. For example, it is not possible for unital Lindbladians because in this case the purity of the state during the dynamical process is monotonically decreasing, regardless of the controls. In this case it is also easy to give an explicit description of all the states reachable by the driven master equation. In the case of affine Lindbladians, the situation is more complicated and "purifications" may occur. What is common in both cases is that the change of purity in the state, and thus the possibility of steering the system out of a sphere of constant purity, is only due to the Lindbladian, not to the coherent controls. For a two-level system, it is shown to what extent this can be used to accomplish closed trajectories which are repeatable in finite time.

The paper is organized as follows. In the next section the model for the Markovian master equation with coherent controls is introduced and its unforced behavior briefly described, in Sec. III the different notions of controllability mentioned above are applied to it, and in Sec. IV the structure of the reachable sets is described from a nonlocal perspective on a few examples of a two-level system. Finally, in Sec. V a few coherent control strategies are discussed, based both on counteracting the dissipation and on actively using it. Part of the material of this work (Sec. III) overlaps with [10], although the presentation is less technical and more oriented to an audience not specialized in control theory. We refer to that work for a more detailed background on geometric control and alternative proofs of the results.

## II. QUANTUM DYNAMICAL SEMIGROUPS WITH COHERENT CONTROLS

The state of a quantum mechanical system in an $N$-dimensional Hilbert space $\mathcal{H}^N$ is described by a positive semidefinite Hermitian operator $\rho$, called the density matrix, having trace $\mathrm{tr}(\rho)=1$ and $\mathrm{tr}(\rho^2)\leqslant 1$. By dimension counting, $\rho$ depends on $n=N^2-1$ real parameters (see [11]). If $\lambda_1,\ldots,\lambda_n$ form a complete orthonormal basis of $N\times N$ traceless Hermitian operators (here the $\lambda_k$ are the so-called Gell-Mann matrices; see Appendix A of Part II of [1] for $N=2,3,4$) and $\lambda_0$ is the rescaled identity matrix, then $\rho=\sum_{j=0}^{n}\mathrm{tr}(\rho\lambda_j)\lambda_j=\sum_{j=0}^{n}\rho_j\lambda_j$, with $\rho_0=N^{-1/2}$ fixed constant and the $n$ real parameters $\rho_j$ giving the parametrization of $\rho$. The vector of expectation values $\boldsymbol{\rho}=[\rho_1\cdots\rho_n]^T$ is called the *coherence vector* of $\rho$ [1]. Due to the constant component along $\lambda_0$, $\boldsymbol{\rho}$ belongs to an affine space characterized by the extra fixed coordinate $\rho_0=N^{-1/2}$. Such $n$-dimensional affine vector $\overline{\boldsymbol{\rho}}=[\rho_0\ \rho_1\cdots\rho_n]^T=[\rho_0\ \boldsymbol{\rho}^T]^T$ is normally referred to as homogeneous coordinates of $\boldsymbol{\rho}$ and lives in a real vector space that has Euclidean inner product given by the trace metric: $\|\overline{\boldsymbol{\rho}}\|=\sqrt{\langle\langle\overline{\boldsymbol{\rho}},\overline{\boldsymbol{\rho}}\rangle\rangle}=\sqrt{\mathrm{tr}(\rho^2)}$. The condition $\mathrm{tr}(\rho^2)\leqslant 1$ then translates in $\overline{\boldsymbol{\rho}}$ space as $\overline{\boldsymbol{\rho}}$ belonging to a subset of the solid affine ball of radius $\sqrt{1-1/N}$ centered at $[\rho_0\ 0\cdots 0]^T$ (call it $\overline{\mathbb{B}}^n$) for all positive times.

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Assume that the Hamiltonian $H$ is composed of a time-invariant part $H_0$ representing the free evolution of the system plus $q$ time-varying forcing terms representing the interaction with $q$ external fields, modeled semiclassically,

$$H(t) = H_0 + \sum_{k=1}^{q} u_k(t) H_k, \quad -iH_0, -iH_k \in \mathfrak{su}(N), \tag{1}$$

with the real parameters $u_k$ representing the amplitude of the control fields applied.

Call $\mathcal{L}_H$ the Hamiltonian part of the dynamics and $\mathcal{L}_D$ the relaxing/dissipating part. In the basis $\{\lambda_j\}$ of traceless Hermitian matrices, the Markovian master equation is expressed as [2]

$$\begin{aligned}\dot{\rho} &= \mathcal{L}_H(\rho) + \mathcal{L}_D(\rho) \\ &= -i\,\mathrm{ad}_H(\rho) + \frac{1}{2}\sum_{j,k=1}^{n} a_{jk}([\lambda_j, \rho\lambda_k] + [\lambda_j\rho, \lambda_k]) \\ &= -i\left[H_0 + \sum_{k=1}^{q} u_k(t) H_k, \rho\right] + \frac{1}{2}\sum_{j,k=1}^{n} a_{jk}(2\lambda_j\rho\lambda_k - \{\lambda_k\lambda_j, \rho\}),\end{aligned} \tag{2}$$

where the Hermitian matrix $A=(a_{jk})$ is positive semidefinite, $A \geqslant 0$, and $\{\cdot,\cdot\}$ is the anticommutator. For the basis $\{\lambda_j\}$, the anticommutator has an affine structure: $\{\lambda_j, \lambda_k\} = (2\sqrt{N}/N)\delta_{jk}\lambda_0 + \sum_{l=1}^{n} d_{jkl}\lambda_l$, with $d_{jkl}$ the real and fully symmetric tensor (with respect to the permutation of any pair of indexes).

The parametrization in terms of the homogeneous coordinates $\overline{\boldsymbol{\rho}}$ corresponds to choosing a matrix representation for the vector fields of Eq. (2). For the Hamiltonian part $\mathcal{L}_H(\cdot)$, this is well known to be simply the Liouville equation in the adjoint representation. The choice of parametrization transforms Eq. (2) into a *control bilinear system* on $\overline{\mathbb{B}^n}$ (see [12–15]). Bilinearity indicates the simultaneous linear dependence from the state $\overline{\boldsymbol{\rho}}$ and the control parameters $u_k$. It provides the simplest possible nonlinear model of a driven quantum-dynamical system. Equation (2) becomes

$$\begin{aligned}\dot{\overline{\boldsymbol{\rho}}} &= \overline{\mathcal{L}_{H_0}}\overline{\boldsymbol{\rho}} + \sum_{k=1}^{q} u_k \overline{\mathcal{L}_{H_k}}\overline{\boldsymbol{\rho}} + \overline{\mathcal{L}_D}\overline{\boldsymbol{\rho}} = \begin{bmatrix} 0 & 0 \\ 0 & -i\,\mathrm{ad}_{H_0}\end{bmatrix}\overline{\boldsymbol{\rho}} \\ &\quad + \sum_{k=1}^{q} u_k \begin{bmatrix} 0 & 0 \\ 0 & -i\,\mathrm{ad}_{H_k}\end{bmatrix}\overline{\boldsymbol{\rho}} + \begin{bmatrix} 0 & 0 \\ \mathbf{v} & \mathbf{L}\end{bmatrix}\overline{\boldsymbol{\rho}}, \quad \overline{\boldsymbol{\rho}} \in \overline{\mathbb{B}^n},\end{aligned} \tag{3}$$

where the last $(n+1)\times(n+1)$ matrix has the following block structure: $\begin{bmatrix} 0 & 0 \\ \mathbf{v} & \mathbf{L}\end{bmatrix} = \sum_{j,k=1}^{n} a_{jk}\overline{L_{jk}}$ with $\overline{L_{jk}} = \begin{bmatrix} 0 & 0 \\ \mathbf{v}_{jk} & L_{jk}\end{bmatrix}$, $j,k=1,\ldots,n$. The $L_{jk} n\times n$ are complex matrices of mixed symmetry and $\mathbf{v}_{jk}$ are imaginary $n$-vectors given by

$$\begin{aligned} L_{jk} &= (L_{jk})_{lr} \\ &= -\frac{1}{4}\sum_{m=1}^{n} [(f_{jmr} + i d_{jmr}) f_{kml} + (f_{kmr} - i d_{kmr}) f_{jml}],\end{aligned} \tag{4}$$

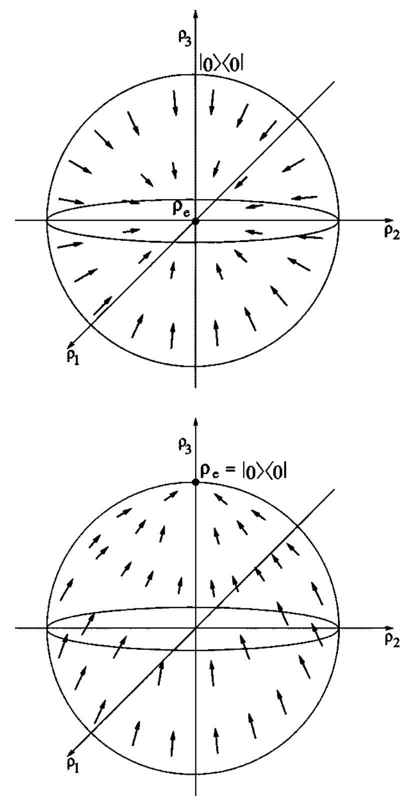


FIG. 1. Cartoons of the vector fields of the unforced evolution for unital $\overline{\mathcal{L}_D}$ (right) and for an affine $\overline{\mathcal{L}_D}$ (here representing a spontaneous emission channel, left).

$$\mathbf{v}_{jk} = \frac{i}{\sqrt{N}}[f_{jk1} \cdots f_{jkn}]^T. \tag{5}$$

In Eqs. (4) and (5), $f_{ljk} = -i(\mathrm{ad}_{\lambda_l})_{jk}$ are the structure constants of the Lie algebra of Hermitian matrices associated with the basis $\lambda_1, \ldots, \lambda_n$.

The behavior of Eq. (3) in the absence of control fields ($u_k = 0$) is well studied and understood [1]. Loosely speaking, since the state space is compact, the effect of the dissipation is to introduce an attractor into the dynamics of the system, as is easy to see for $N=2$ on the Bloch ball. In this case, a number of characteristic dissipation channels is described in [16] (their infinitesimal generators are given, for example, in [10]). Typically one distinguishes between unital (when $\mathbf{v} = \mathbf{0}$) and affine Lindbladians. The qualitative difference between the two unforced dynamics is depicted in Fig. 1 where the vector fields for different initial conditions are shown for a unital Lindbladian (e.g., a combination of bit flip and phase flip channels) on the left, and, on the right, for an amplitude damping channel, i.e., for the model of a two-level atom with spontaneous emission **(**Example (3) in Sec. V of [10]; see also Sec. V below**)**. As can be seen in Fig. 1, the main qualitative difference between the two cases is in the different location of the equilibrium point. When such equilibrium point is independent of the initial condition [the "genuinely relaxing semigroup" condition (A3) below] then then we have a global attractor for Eq. (3). While for a unital $\overline{\mathcal{L}_D}$ the fixed point is always the completely random state, in correspondence with an affine $\overline{\mathcal{L}_D}$ the equilibrium may be placed everywhere in the Bloch ball, thus allowing for a more variegated behavior in the dynamical semigroup. In both cases a basic task of a control action would be to counteract dissipation, i.e., to "go against" the irreversibility induced by the asymptotically stable character of the equilibrium point. A more sophisticated task would be to do this while accomplishing also a desired state transfer. In order to gain insight into these problems, it is useful to carry out first a controllability analysis for Eq. (3).

## III. REACHABLE SETS AND CONTROLLABILITY NOTIONS

The starting point of a controllability investigation is usually an analysis of the reachable set, i.e., of the set of states $\overline{\boldsymbol{\rho}}$ in $\overline{\mathbb{B}^n}$ that can be reached by the dynamics (3) by means of all possible coherent controls starting from a given initial condition $\overline{\boldsymbol{\rho}_i}$. When the controls are allowed to vary in a suitable class of functions (piecewise constant is enough for our purposes), this functional analysis not only reveals if the entire state space can be explored by suitable excitation (controllability), but is also important to device a control-oriented notion of *reversibility* of the integral curves of Eq. (3) in both a local and nonlocal sense. To this aim, we will introduce the concepts (standard in control theory, see e.g., [14]) of small-time controllability and finite-time controllability.

In the following, rather than treating $\overline{\mathcal{L}_D}$ as a disturbance, we will assume we are dealing only with a precisely known value of $A$ and hence of $\overline{\mathcal{L}_D}$.

We make the following assumptions.

(A1) If $A=0$ the system (3) is controllable.

(A2) The parameters $a_{jk}$, $j,k=1,\ldots,n$, are fixed and known exactly.

(A3) The unital part $\mathbf{L}$ of the dissipation is such that $\mathbf{L}^{-1}$ always exists.

Assumption (A3) simply means that the unforced dynamics has an equilibrium point independent of the initial condition, $\rho_e=-(-i\,\mathrm{ad}_{H_0}+\mathbf{L})^{-1}\mathbf{v}\rho_0$, and, as mentioned above, is normally referred to as a "genuinely relaxing" semigroup [1]. The case where Assumption (A1) holds is obviously the most interesting one: if (A1) is not satisfied then little can be said for the controlled master equation. As a consequence of (A2), we can treat $\overline{\mathcal{L}_D}$ as a part of the drift term (together with $\overline{\mathcal{L}_{H_0}}$). In control terminology, the *drift* is a vector field which does not depend on any control parameter. Looking at Eq. (3), in particular, it can be thought of as the part of the infinitesimal generator which is not directly reversible by acting on the control inputs and as such it gives the semigroup structure discussed above.

Given $\overline{\boldsymbol{\rho}_i}\in\overline{\mathbb{B}^n}$, let us call $\mathcal{R}(\overline{\boldsymbol{\rho}_i},T)$ the *reachable set from* $\overline{\boldsymbol{\rho}_i}$ *at time* $T>0$ for the system (3), i.e., the set of $\overline{\boldsymbol{\rho}}\in\overline{\mathbb{B}^n}$ such that $\overline{\boldsymbol{\rho}}(0)=\overline{\boldsymbol{\rho}_i}$ and $\overline{\boldsymbol{\rho}}(T)=\overline{\boldsymbol{\rho}}$, $T>0$, for some admissible control $u_1,\ldots,u_q$. If $\mathcal{R}(\overline{\boldsymbol{\rho}_i},\leqslant T)=\cup_{0\leqslant t\leqslant T}\mathcal{R}(\overline{\boldsymbol{\rho}_i},t)$, then the reachable set from $\overline{\boldsymbol{\rho}_i}$ is $\mathcal{R}(\overline{\boldsymbol{\rho}_i})=\cup_{0\leqslant t\leqslant\infty}\mathcal{R}(\overline{\boldsymbol{\rho}_i},t)$. We make use of the following control-theoretical notions: (1) accessibility, (2) small-time controllability, (3) finite-time controllability, (4) controllability. They correspond respectively to (see, e.g., [17]) (1) $\dim(\mathcal{R}(\overline{\boldsymbol{\rho}_i},\leqslant T))=n$ $\forall\ T>0$, (2) $\overline{\boldsymbol{\rho}_i}\in\mathrm{int}\mathcal{R}(\overline{\boldsymbol{\rho}_i},T)$ $\forall\ T>0$, (3) for a given $T_f>0$, $\overline{\boldsymbol{\rho}_f}\in\mathrm{int}\mathcal{R}(\overline{\boldsymbol{\rho}_i},\leqslant T_f)$ $\forall\ \overline{\boldsymbol{\rho}_i},\overline{\boldsymbol{\rho}_f}\in\overline{\mathbb{B}^n}$, and (4) $\overline{\boldsymbol{\rho}_f}\in\mathrm{cl}\mathcal{R}(\overline{\boldsymbol{\rho}_i})$ $\forall\ \overline{\boldsymbol{\rho}_i},\overline{\boldsymbol{\rho}_f}\in\overline{\mathbb{B}^n}$, where $\mathrm{int}(\cdot)$ means interior and $\mathrm{cl}(\cdot)$ closure. The accessibility property expresses the fact that by varying the controls, the reachable sets are open and dense in the state space $\overline{\mathbb{B}^n}$. The accessibility condition disregards the difference between the (nonreversible) drift $(\overline{\mathcal{L}_{H_0}}+\overline{\mathcal{L}_D})\overline{\boldsymbol{\rho}}$ and the (reversible) control vector fields $\overline{\mathcal{L}_{H_1}}\overline{\boldsymbol{\rho}},\ldots,\overline{\mathcal{L}_{H_q}}\overline{\boldsymbol{\rho}}$ and is just concerned with testing the dimension of the orbits of the system when varying the control inputs regardless of their effective reversibility. However, accessibility is only a necessary condition for controllability, i.e., it does not say anything about the controls enabling arbitrary and reversible manipulation of the state $\overline{\boldsymbol{\rho}}$. In terms of reachable states, $\mathcal{R}(\overline{\boldsymbol{\rho}_i},\leqslant T)$ open in $\overline{\mathbb{B}^n}$ but not containing $\overline{\boldsymbol{\rho}_i}$ in its interior implies that it is not possible to reach in time $T$ an arbitrary neighborhood of $\overline{\boldsymbol{\rho}_i}$ by means of any possible control function. For the system (3) with $\overline{\mathcal{L}_D}\neq 0$ this will always be the case: the effect of the drift can never be suppressed by means of the control vector fields. Depending on the structure of $\overline{\mathcal{L}_D}$ it may be possible to achieve controllability only as a limit process as $T\to\infty$. Physically this implies that (3) is never reversible in "small time," although for a time long enough (sometimes $\infty$) a process may be reversed and a particular state $\overline{\boldsymbol{\rho}_f}$ reached.

A fundamental difference between (1) and (2)–(4) above is that the accessibility concept naturally admits an equivalent infinitesimal characterization, the so-called Lie algebraic rank condition (LARC), which affirms that a system like (3) is accessible if and only if the Lie algebra generated by the vector fields $(\overline{\mathcal{L}_{H_0}}+\overline{\mathcal{L}_D})\overline{\boldsymbol{\rho}}$, $\overline{\mathcal{L}_{H_1}}\overline{\boldsymbol{\rho}},\ldots,\overline{\mathcal{L}_{H_q}}\overline{\boldsymbol{\rho}}$ has dimension $n$. This characterization is purely algebraic and as such easy to check. If we call $\mathfrak{g}=\mathrm{Lie}(\overline{\mathcal{L}_{H_0}}+\overline{\mathcal{L}_D},\overline{\mathcal{L}_{H_1}},\ldots,\overline{\mathcal{L}_{H_q}})$ the corresponding Lie algebra of matrices, for our Markovian master equation we also have the following condition.

*Theorem 1*. The system (3) is accessible if and only if $\mathfrak{g}$ is transitive on $\mathbb{R}^n$.

*Proof*. For matrix systems like (3), the easiest way to check LARC is to test the transitivity of $\mathfrak{g}$ in $\mathbb{R}^n$ (recall that $\overline{\mathbb{B}^n}$ is an affine ball in $\mathbb{R}^n$). This is a well-studied topic in geometric control [18]. ■

At the level of the corresponding group of transformations $\exp(\mathfrak{g})$, if one disregards the difference between drift and control vector fields (as the property of accessibility does) then transitivity implies $\exp(\mathfrak{g})\mathbb{R}^n=\mathbb{R}^n$ and corresponds exactly to our controllability notion. Under Assumption (A1), the Hamiltonian vector fields of (1) form the Lie algebra $\mathfrak{su}(N)$ of dimension $n$. The $N\times N$ traceless skew-Hermitian matrices $-i\lambda_1,\ldots,-i\lambda_n$ are a basis of $\mathfrak{su}(N)$. In the adjoint representation, $-i\lambda_1,\ldots,-i\lambda_n$ are mapped into the $n\times n$ real and skew-symmetric matrices $-i\,\mathrm{ad}_{\lambda_1},\ldots,-i\,\mathrm{ad}_{\lambda_n}$ which form a basis of the Lie algebra $\mathrm{ad}_{\mathfrak{su}(N)}$, subalgebra of $\mathfrak{so}(n)$ (proper for $N>2$). This would be the Lie algebra of the corresponding Liouville equation. However, the dissipation term $\bar{\mathcal{L}}_D$ is not coherent and as such it enlarges the integral group of (3) from $\exp(\mathrm{ad}_{\mathfrak{su}(N)})$ to one of the Lie groups properly containing it. Thus the Lie algebras of interest here must (1) be real, (2) properly contain $\mathrm{ad}_{\mathfrak{su}(N)}$, and (3) be transitive on $\mathbb{R}^n$. Such Lie algebras are for example $\mathfrak{sl}(n)$, $\mathfrak{gl}(n)$ and their semidirect extensions $\mathfrak{sl}(n)Ⓢ\mathbb{R}^n$, $\mathfrak{gl}(n)Ⓢ\mathbb{R}^n$. But $\mathfrak{sl}(n)$ and $\mathfrak{sl}(n)Ⓢ\mathbb{R}^n$ are not admissible, as they are not compatible with the assumption of $A \geqslant 0$. See [19] for a complete list of all Lie algebras transitive on $\mathbb{R}^n$.

The condition of Theorem 1 is generically verified for $\bar{\mathcal{L}}_D \neq 0$, i.e., almost all $\bar{\mathcal{L}}_D$ are such that $\mathfrak{g}=\mathfrak{gl}(n)$ (if $\bar{\mathcal{L}}_D$ unital) or $\mathfrak{g}=\mathfrak{gl}(n)Ⓢ\mathbb{R}^n$ (if $\bar{\mathcal{L}}_D$ is affine). An exception occurs, for example, when $N>2$ and the unital part of $\bar{\mathcal{L}}_D$ is diagonal. In fact, $\bar{\mathcal{L}}_D$ diagonal (and unital) belongs to $\mathrm{span}(I)$ $=\mathfrak{gl}(n)\backslash\mathfrak{sl}(n)$. Therefore it commutes with $\bar{\mathcal{L}}_{H_1},\ldots,\bar{\mathcal{L}}_{H_q}$ and the controls cannot generate new directions of motion. Similarly, when the unital part of $\bar{\mathcal{L}}_D$ belongs to $\mathfrak{so}(n)\oplus\mathrm{span}(I)$: the diagonal part commutes with the control vector fields while Lie brackets of $\bar{\mathcal{L}}_{H_1},\ldots,\bar{\mathcal{L}}_{H_q}$ with the $\mathfrak{so}(n)$ part cannot exit the compact subalgebra $\mathfrak{so}(n)$. Notice, however, that for $N=2$ $\mathrm{ad}_{\mathfrak{su}(2)}=\mathfrak{so}(3)$ and $\mathfrak{so}(3)\oplus\mathrm{span}(I)$ is the Lie algebra of homoteties of $\mathbb{R}^3$ whose action is transitive on $\mathbb{R}^3$. When $\bar{\mathcal{L}}_D \notin \mathfrak{so}(n)\oplus\mathrm{span}(I)$ (or, again, its semidirect extension if $\bar{\mathcal{L}}_D$ is not unital) then $\bar{\mathcal{L}}_D$ must have a noncompact semisimple component and we can use a known theorem affirming that the set of pairs of vector fields in a semisimple Lie algebra that generate the entire Lie algebra by means of repeated Lie brackets is open and dense in the Lie algebra itself; see, e.g., Theorem 12, Chapter 6 of [14]. $\bar{\mathcal{L}}_D \notin \mathfrak{so}(n)$ $\oplus\,\mathrm{span}(I)$ is a generic condition in $\mathfrak{gl}(n)$, hence, for almost all $\bar{\mathcal{L}}_D$ we have that $\mathfrak{g}$ is equal to $\mathfrak{gl}(n)$ or $\mathfrak{gl}(n)Ⓢ\mathbb{R}^n$ and thus we have proved the following:

*Corollary 1*. The system (3) is generically accessible.

Unlike accessibility, the testing of controllability conditions is for general nonlinear systems a more complicated matter. Fortunately, for the quantum Markovian master equation the bilinearity of the vector fields and the peculiar structure of (3) given by the complete positivity assumption greatly simplify the task, as we will see. In general, as mentioned above, the complications come from the fact that the initial condition lies on the boundary of the reachable set, rather than in its interior. In fact, this has as consequence that some neighborhoods of the initial condition are not reachable by the control action and thus the system is not controllable. For the system (3) this is always the case when the drift term $\bar{\mathcal{L}}_{H_0}+\bar{\mathcal{L}}_D$ is non-null and nonunitary.

*Theorem 2*. The system (3) is neither small-time nor finite-time controllable.

*Proof*. Consider first the small-time controllability property. Assume the contrary holds, i.e., that $\bar{\boldsymbol{\rho}}_i \in \mathrm{int}\mathcal{R}(\bar{\boldsymbol{\rho}}_i,T)$ $\forall\ T>0$. Then any small enough neighborhood $\mathcal{N}(\bar{\boldsymbol{\rho}}_i)$ of $\bar{\boldsymbol{\rho}}_i$ is such that $\mathcal{N}(\bar{\boldsymbol{\rho}}_i)\subset\mathrm{int}\mathcal{R}(\bar{\boldsymbol{\rho}}_i,T)$. Consider the case of initial condition which is a mixed state: $0<\|\bar{\boldsymbol{\rho}}_i\|^2<1$. Then $\bar{\boldsymbol{\rho}}_i$ $\in \mathrm{int}\mathcal{R}(\bar{\boldsymbol{\rho}}_i,T)$ $\forall\ T>0$ implies that $\exists$ a neighborhood $\mathcal{N}(\bar{\boldsymbol{\rho}}_i)$ and control inputs such that states $\bar{\boldsymbol{\rho}}_{f_1}(T)\in\mathcal{N}(\bar{\boldsymbol{\rho}}_i)$ with $\|\bar{\boldsymbol{\rho}}_{f_1}(T)\|^2>\|\bar{\boldsymbol{\rho}}_i\|^2$ and $\bar{\boldsymbol{\rho}}_{f_2}(T)\in\mathcal{N}(\bar{\boldsymbol{\rho}}_i)$ with $\|\bar{\boldsymbol{\rho}}_{f_2}(T)\|^2<\|\bar{\boldsymbol{\rho}}_i\|^2$ are reachable $\forall\ T>0$. But the derivative of the function $\|\bar{\boldsymbol{\rho}}\|^2$

$$
\begin{aligned}
\frac{d}{dt}\|\bar{\boldsymbol{\rho}}\|^2 &= 2\langle\langle\bar{\boldsymbol{\rho}},\dot{\bar{\boldsymbol{\rho}}}\rangle\rangle = 2[\langle\langle\bar{\boldsymbol{\rho}},(\bar{\mathcal{L}}_{H_0}+\bar{\mathcal{L}}_D)\bar{\boldsymbol{\rho}}\rangle\rangle \\
&\quad + u_1\langle\langle\bar{\boldsymbol{\rho}},\bar{\mathcal{L}}_{H_1}\bar{\boldsymbol{\rho}}\rangle\rangle+\cdots+u_q\langle\langle\bar{\boldsymbol{\rho}},\bar{\mathcal{L}}_{H_q}\bar{\boldsymbol{\rho}}\rangle\rangle] \\
&= 2\langle\langle\bar{\boldsymbol{\rho}},\bar{\mathcal{L}}_D\bar{\boldsymbol{\rho}}\rangle\rangle,
\end{aligned}
\tag{6}
$$

shows that the variation of purity of $\bar{\boldsymbol{\rho}}_i$ cannot be altered *locally* by the control action and is determined only by the direction of $\bar{\mathcal{L}}_D$: if $\bar{\mathcal{L}}_D$ points inward on the sphere $\|\bar{\boldsymbol{\rho}}_i\|^2$ then the mixing locally increases, if it points outward then the state is locally purified. Hence $\bar{\boldsymbol{\rho}}_{f_1}(T)$ and $\bar{\boldsymbol{\rho}}_{f_2}(T)$ cannot be both reached in any time $T$ and we have a contradiction.

Concerning finite-time controllability, from Eq. (6) it is clear that the change in purity occurs only because of $\bar{\mathcal{L}}_D$ even on a nonlocal basis. Depending on the value of $\bar{\mathcal{L}}_D$, the uncontrolled equation

$$
\dot{\bar{\boldsymbol{\rho}}}=(\bar{\mathcal{L}}_{H_0}+\bar{\mathcal{L}}_D)\bar{\boldsymbol{\rho}}
\tag{7}
$$

will or will not have an equilibrium point and the flow of Eq. (7) will or will not cross all the "purity level surfaces" in $\mathbb{B}^n$ while approaching it. According to our definition, for finite-time controllability to hold, it has to hold for all $\bar{\boldsymbol{\rho}}_i$ and $\bar{\boldsymbol{\rho}}_f$ in $\mathbb{B}^n$. In general if in Eq. (7) $\lim_{t\to\infty}\bar{\boldsymbol{\rho}}(t)=\bar{\boldsymbol{\rho}}_e$ **(**even perhaps depending on the initial condition, $\bar{\boldsymbol{\rho}}_e=\bar{\boldsymbol{\rho}}_e(\bar{\boldsymbol{\rho}}_i)$**)** with $\|\bar{\boldsymbol{\rho}}_e\|^2$ $<1$ then controllability does not hold at all. To check it just consider $\bar{\boldsymbol{\rho}}_i$ such that $\|\bar{\boldsymbol{\rho}}_i\|^2\leqslant\|\bar{\boldsymbol{\rho}}_e\|^2$. Then at most the ball of radius $\|\bar{\boldsymbol{\rho}}_e\|$ is reachable. If instead $\|\bar{\boldsymbol{\rho}}_e\|^2=1$ then controllability is only asymptotic. In fact, the reachable set in finite time is at most a closed set contained inside the open set $\{\bar{\boldsymbol{\rho}}$ such that $\|\bar{\boldsymbol{\rho}}\|^2<1\}$ and only asymptotically may $\mathrm{cl}\mathcal{R}(\bar{\boldsymbol{\rho}}_i)$ become equal to $\mathbb{B}^n$. ■

In the proof above, we excluded the case that state transfer can occur while maintaining the same purity. Of course, from a practical point of view, if we use strong pulses or if the decay time induced by the $\bar{\mathcal{L}}_D$ is long enough, then, in the first approximation, state transfer between states belonging to the same purity sphere in $\mathbb{B}^n$ can occur. Still, "purifications" of $\bar{\boldsymbol{\rho}}$ are possible only through $\bar{\mathcal{L}}_D$ and thus the impossibility of controllability in finite time whenever $\|\bar{\boldsymbol{\rho}}\|^2<1$.

*Remark 1*. It is worth emphasizing the meaning of the controllability concepts introduced above.

(i) The lack of reversibility normally associated with the quantum Markovian master equation is captured by the lack of small-time controllability.

(ii) The lack of finite-time controllability could be expressed as the impossibility of accomplishing an arbitrary cyclic trajectory in finite time (with 100% probability) by means of coherent control alone.

## IV. GLOBAL STRUCTURE OF THE REACHABLE SETS

Although controllability in small time and finite time is missing, it is possible to give a detailed global description of

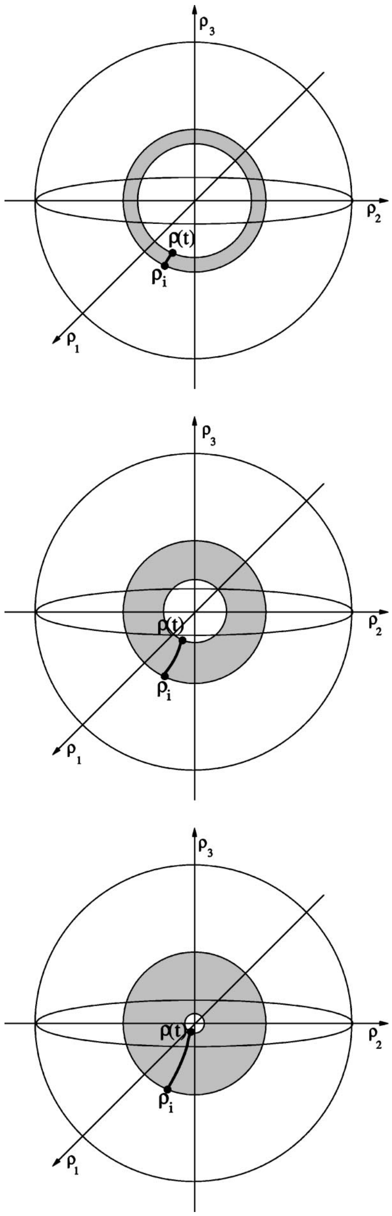


FIG. 2. Cartoons of the reachable sets $\mathcal{R}(\overline{\boldsymbol{\rho}}_i, \leqslant t)$ (gray areas) for the two-level atom with phase damping ($\overline{\mathcal{L}_D}$ unital) as $t$ grows.

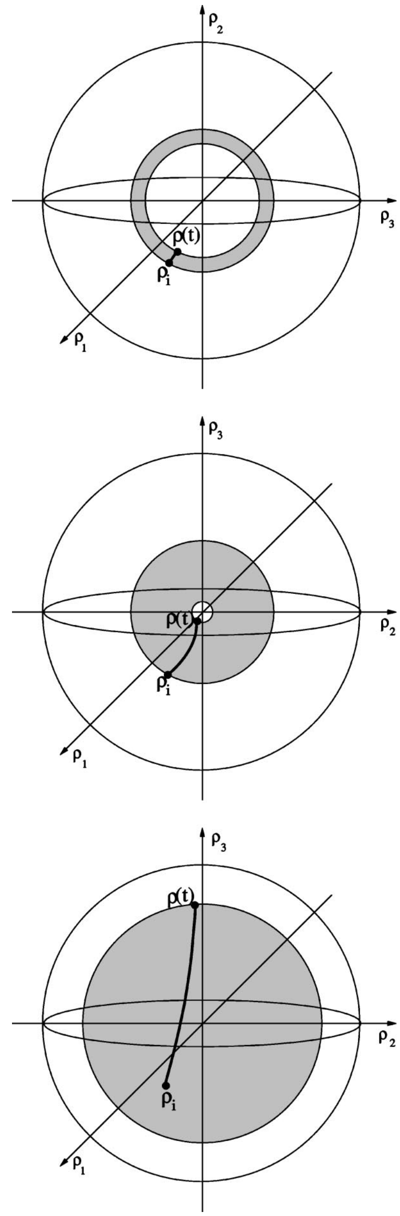


FIG. 3. Cartoons of the reachable sets $\mathcal{R}(\overline{\boldsymbol{\rho}}_i, \leqslant t)$ (gray areas) for the two-level atom with spontaneous emission ($\overline{\mathcal{L}_D}$ affine) as $t$ grows.

the sets of states reachable from a given initial condition $\overline{\boldsymbol{\rho}}_i$. The simplest case is when $\overline{\mathcal{L}_D}$ is unital.

*Proposition 1*. If $\overline{\mathcal{L}_D}$ unital and (3) accessible, then $\mathcal{R}(\overline{\boldsymbol{\rho}}_i, \leqslant t)$ is an annulus of inner and outer radius, respectively, $\|\overline{\boldsymbol{\rho}}(t)\|$ and $\|\overline{\boldsymbol{\rho}}_i\|$. When $t \to \infty$, $\mathrm{cl}\mathcal{R}(\overline{\boldsymbol{\rho}}_i)$ is the ball of radius $\|\overline{\boldsymbol{\rho}}_i\|$.

*Proof*. In Eq. (6), $\overline{\mathcal{L}_D}$ unital can point only inward: $(d/dt)\|\overline{\boldsymbol{\rho}}\|^2 \leqslant 0 \quad \forall\ t \geqslant 0$. Furthermore, accessibility implies that $\overline{\boldsymbol{\rho}}(t)$ can be placed on any point of the sphere of radius $\|\overline{\boldsymbol{\rho}}(t)\|$, regardless of the existence of a fixed point for $\overline{\mathcal{L}_D}$. Thus we have the form of an annulus at time $t$ for $\mathcal{R}(\overline{\boldsymbol{\rho}}_i, \leqslant t)$ and the convergence to the center $[\rho_0\ 0\ 0\ 0]^T$ as $t \to \infty$. ■

Figure 2 shows a sketch of how the reachable sets grow monotonically in time when $\overline{\mathcal{L}_D}$ is a phase damping operator (Example (2) of Sec. V of [10]; see also Sec. V below), interplaying with the controls. If we measure the purity of a density operator by the norm $\|\overline{\boldsymbol{\rho}}\|$, then we automatically get the following.

*Corollary 2*. For $\overline{\mathcal{L}_D}$ unital, the purity of $\overline{\boldsymbol{\rho}}$ subject to (3) is nonincreasing.

*Proof*. It follows again from $(d/dt)\|\overline{\boldsymbol{\rho}}\|^2 \leqslant 0\ \forall\ t \geqslant 0$. ■

The situation is more complicated when $\overline{\mathcal{L}_D}$ is affine. In fact, in this case, the monotonicity property may not hold any more, depending on the values of $\overline{\boldsymbol{\rho}}_i$ and $\overline{\mathcal{L}_D}$. A typical example of what can happen is shown in Fig. 3 for the two-level system with spontaneous emission. In this case, the corresponding Lindbladian operator is affine and has a fixed equilibrium point at the ground state $\overline{\boldsymbol{\rho}_e}=[\rho_0\ 0\ 0\ 1/\sqrt{2}]^T$ (i.e., $\rho_e=|0\rangle\langle 0|$).

## V. COHERENT CONTROL STRATEGIES IN THE PRESENCE OF DISSIPATION

In the first part of this section we discuss how to counteract dissipation by means of coherent control; in the last part how to actively use it for the purposes of state transfer. When $\overline{\mathcal{L}_D}$ is unital, Corollary 2 states a monotonicity property of the Hilbert-Schmidt norm of $\overline{\boldsymbol{\rho}}$ that the coherent control cannot eliminate. However, such a control action can be used to modify the rate of decay due to the dissipation (at least in the case when $\mathbf{L}$ is not proportional to the identity), by placing $\overline{\boldsymbol{\rho}}$ along the slowest direction of decay.

For example, if we have the phase damping channel of a two-level system

$$\dot{\overline{\boldsymbol{\rho}}}=h_{0_3}\overline{M_3}\overline{\boldsymbol{\rho}}+\sum_{k=1}^{3}u_k\overline{M_k}\overline{\boldsymbol{\rho}}+c\begin{bmatrix}0&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&0\end{bmatrix}\overline{\boldsymbol{\rho}}$$

with $c>0$, it is enough to place the state in $[\rho_0\ 0\ 0\ \rho_3]^T$ to get $(d/dt)\|\overline{\boldsymbol{\rho}}\|^2=0$ in Eq. (6). This tells that we have a quantum channel not satisfying Assumption (A3). In fact, under Assumption (A3) any unital $\overline{\mathcal{L}_D}$ is such that

$$\max_{\overline{\boldsymbol{\rho}}}\langle\langle\overline{\boldsymbol{\rho}},\overline{\mathcal{L}_D\boldsymbol{\rho}}\rangle\rangle<0. \tag{8}$$

The proof follows from $\langle\langle\overline{\boldsymbol{\rho}},\overline{\mathcal{L}_D\boldsymbol{\rho}}\rangle\rangle=\langle\langle\boldsymbol{\rho},\mathbf{L}\boldsymbol{\rho}\rangle\rangle$ and the positive definiteness of $A$. From Eq. (6), one has that the control is not entering into $(d/dt)\|\overline{\boldsymbol{\rho}}\|^2$ and hence in Eq. (8). The objective of an "optimal" control strategy aiming at rejecting as much as possible the dissipation should be to drive the state to the maximizing $\overline{\boldsymbol{\rho}}$ as fast as possible. Because of the linearity, the property of maximizing (8) is shared by an entire "ray" of Bloch vectors. Hence, once a strong pulse has driven the state to the maximizing $\overline{\boldsymbol{\rho}}$ it can be switched off. In the case of a depolarizing channel **(**Example (1) in Sec. V of [10]**)**, each direction has the same dissipation rate, and therefore the coherent control is totally useless. A number of related results on control of unital Lindbladians appear in a recent paper [20].

When instead $\overline{\mathcal{L}_D}$ is affine, we can have a more constructive use of the coherent controls for the purposes of rejecting the dissipation because $\overline{\mathcal{L}_D}$ is not always radial and its nonradial (i.e., tangential and hence unitary) part can be suppressed by a suitably chosen time-varying control law. Rewriting Eq. (3) as

$$\dot{\boldsymbol{\rho}}=-i\left(\mathrm{ad}_{H_0}+\sum_{k=1}^{q}u_k\mathrm{ad}_{H_k}\right)\boldsymbol{\rho}+\boldsymbol{L}\boldsymbol{\rho}+\boldsymbol{v}\rho_0,$$

one possible choice of the control law $u_k=u_k(\boldsymbol{\rho})$ is given by the following algebraic (state dependent) constraints:

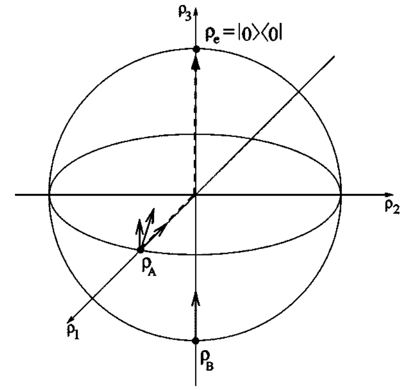


FIG. 4. Solid arrow: dissipation operator in two different states (in $\boldsymbol{\rho}_A$ it is decomposed into radial and tangential components) for the two-level spontaneous emission model. Dashed line: possible "controlled" trajectory.

$$-i\left(\mathrm{ad}_{H_0}+\sum_{k=1}^{q}u_k\mathrm{ad}_{H_k}\right)\boldsymbol{\rho}+\boldsymbol{v}\rho_0=0. \tag{9}$$

This may or may not have a solution, depending on the number of controls available, on the structure of the dissipation, and on the state $\boldsymbol{\rho}$ in which it is computed.

It is convenient to show what is happening on an example. Consider the "amplitude damping channel," i.e., the two-level system with spontaneous emission mentioned above. Its differential equation is given by

$$\dot{\boldsymbol{\rho}}=\begin{bmatrix}0&-h_{0_3}-u_3&u_2\\h_{0_3}+u_3&0&-u_1\\-u_2&u_1&0\end{bmatrix}\boldsymbol{\rho}+\gamma_{\downarrow}\begin{bmatrix}-\frac{1}{2}&0&0\\0&-\frac{1}{2}&0\\0&0&-1\end{bmatrix}\boldsymbol{\rho}+\gamma_{\downarrow}\begin{bmatrix}0\\0\\1\end{bmatrix}\rho_0, \tag{10}$$

were we assume all three controls are available and, for the sake of simplicity, that $H_0=0$. The condition (9) becomes

$$\begin{aligned}-u_3\rho_2+u_2\rho_3&=0,\\u_3\rho_1-u_1\rho_3&=0,\\-u_2\rho_1+u_1\rho_2+\gamma_{\downarrow}\rho_0&=0.\end{aligned} \tag{11}$$

For example, in $\boldsymbol{\rho}_A=[\rho_1\ 0\ 0]^T$ (see Fig. 4), the affine term behaves as unitary and can be completely eliminated by a single control $u_2=\gamma_{\downarrow}\rho_0/\rho_1$. On the contrary, in $\boldsymbol{\rho}_B=[0\ 0\ \rho_3]^T$ the affine term is "purely" radial and no coherent control can eliminate it (since $\rho_1$ and $\rho_2$ are 0). Even when $\mathbf{v}\rho_0$ can be suppressed, as in $\boldsymbol{\rho}_A$, nothing can be done against the unital part $\mathbf{L}\boldsymbol{\rho}$ which still tends to steer the system to the completely random state. In the "ideal" case, the strategy above gives the dashed trajectory in Fig. 4.

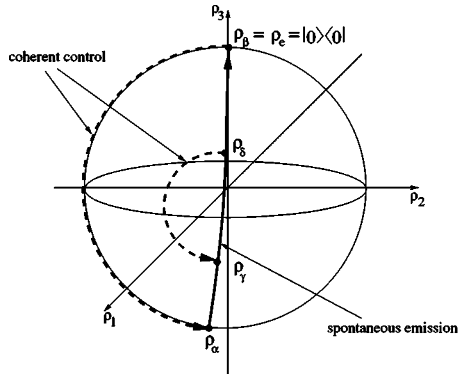


FIG. 5. Cartoon of the cyclic trajectories on the Bloch sphere for a two-level atom with coherent control and spontaneous emission.

Alternatively, one could design $u_k = u_k(\boldsymbol{\rho})$ by means of an optimal control problem aiming at keeping $\boldsymbol{\rho}$ away from $\boldsymbol{\rho}_e$. Since $\boldsymbol{\rho}_e$ is known, instead of $\|\overline{\boldsymbol{\rho}}\|$ one should consider $\|\boldsymbol{\rho} - \boldsymbol{\rho}_e\|$ as the rate of decay due to the dissipation in the cost function (together with the norm of the control action). Notice from the derivative

$$\frac{d}{dt}\|\boldsymbol{\rho} - \boldsymbol{\rho}_e\|^2 = 2\langle\langle \boldsymbol{\rho} - \boldsymbol{\rho}_e, \dot{\boldsymbol{\rho}}\rangle\rangle$$

$$= 2\Bigg[\langle\langle \boldsymbol{\rho}, \boldsymbol{L}\boldsymbol{\rho} + \boldsymbol{v}\rho_0\rangle\rangle - \left\langle\!\!\left\langle \boldsymbol{\rho}_e, -i\left(\mathrm{ad}_{H_0} + \sum_{k=1}^{q} u_k \mathrm{ad}_{H_k}\right)\boldsymbol{\rho}\right\rangle\!\!\right\rangle - \langle\langle \boldsymbol{\rho}_e, \boldsymbol{L}\boldsymbol{\rho} + \boldsymbol{v}\rho_0\rangle\rangle\Bigg]$$

how this time the control plays indeed a nontrivial role.

As explained in [9], a nonzero steady state control can be used also to modify the equilibrium point $\boldsymbol{\rho}_e = -[-i(\mathrm{ad}_{H_0} + \sum_{k=1}^{q}\mathrm{ad}_{H_k}) + \mathbf{L}]^{-1}\mathbf{v}\rho_0$, moving it from a pure to a mixed state.

The bottom line is that regardless of the control law and even if we can modify the equilibrium point, we cannot modify the character of local irreversibility due to the lack of small-time controllability (see Remark 1). However, if we decide to use decoherence, rather than just trying to suppress it "locally," then it is for example possible to accomplish repeatable tasks. From the proof of Theorem 2, we have that cyclic trajectories involving only mixed states (even with different degrees of mixing) may be feasible using coherent controls if $\overline{\mathcal{L}_D}$ is affine, and could be accomplished in finite time. If instead they involve pure states (and thus "complete purification") they can only occur in infinite time even if $\overline{\mathcal{L}_D}$ is affine. For example, for the two-level atom with coherent controls and spontaneous emission, a closed cycle is described in Fig. 5 and in Table I.

TABLE I. Cyclic trajectories of Fig. 5 for a two level-atom with coherent control and spontaneous emission.

| State transfer | Induced by | Time required |
|---|---|---|
| $\boldsymbol{\rho}_\beta \to \boldsymbol{\rho}_\alpha$ | Coherent control | Finite |
| $\boldsymbol{\rho}_\alpha \to \boldsymbol{\rho}_\beta$ | Spontaneous emission | Infinite |
| $\boldsymbol{\rho}_\delta \to \boldsymbol{\rho}_\gamma$ | Coherent control | Finite |
| $\boldsymbol{\rho}_\gamma \to \boldsymbol{\rho}_\delta$ | Spontaneous emission | Finite |

## VI. CONCLUSION AND OUTLOOK

The main advantage of the control-theoretic formalism presented in this paper is that it allows us to give a more "fine-graded" characterization of the notion of irreversibility normally associated to dissipating quantum dynamical semigroups and of how it interacts with the coherent control fields. This shows the intrinsic limits of unitary control for these systems. In particular, the impossibility of fully rejecting the dissipation is captured in control-theoretic terms by the lack of small-time controllability. While this does not forbid the construction of motion planning strategies that make "active" use of the dissipation,[1] it also naturally calls for richer classes of control fields than just unitary ones to be studied. Several potential candidates have already been discussed in the literature for different physical settings, like the use of gradient fields in NMR [21] and the use of quantum feedback in quantum optics [22]. In the first case, varying the constant longitudinal magnetic field, a nonunitary global effect is obtained and can be used for control purposes; in the second case a nonunitary degree of freedom is provided by the back action effect of a weak measurement.

Another potential direction of investigation is to look for "controlled decoherence free subspaces," i.e., for particular (unitary) control design able to actively confine decoherence to a particular subspace of the state space, combining the decoherence-free techniques of widespread use in quantum computing with the "disturbance decoupling" methods of nonlinear control theory (see Chap. 7 of [23] for an overview).

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[1]Most of the techniques for state transfer in the presence of relaxation in the NMR practice are based on this principle; see [24,25].

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