Controllability and simultaneous controllability of isospectral bilinear control systems on complex flag manifolds

Claudio Altafini
SISSA-ISAS, International School for Advanced Studies
via Beirut 2-4, 34014 Trieste, Italy,
tel. +39 040 3787431, fax + 39 040 3787528

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Abstract

For isospectral bilinear control systems evolving on the so-called complex flag manifolds (i.e., on the orbits of the Hermitian matrices under unitary conjugation action) it is shown that controllability is almost always verified. Easy and generic sufficient conditions are provided. The result applies to the problem of density operator controllability of finite dimensional quantum mechanical systems. In addition, we show that systems having different drifts (corresponding for example to different Larmor frequencies) are simultaneously controllable by the same control field.

keyword Controllability, Simultaneous controllability, Bilinear control systems, Isospectral evolution, Quantum control.

1 Introduction

The conjugation action of the unitary group over the vector space of Hermitian matrices generates a family of compact, connected orbits, called complex flag manifolds [1, 2], that foliate the Hermitian matrices into equivalence classes having as a complete set of invariants the eigenvalues of an Hermitian matrix representative. These manifolds have different dimensions according to the number of distinct eigenvalues and to their multiplicities. In this work we consider bilinear matrix control systems evolving on such complex flag manifolds. Since the eigenvalues of the state matrix are constant, the evolution must be isospectral.

We show that almost every such isospectral system is controllable, i.e., that controllability is a generic property. This can easily be deduced from i) the genericity of controllability on the unitary group; ii) the transitivity of the conjugation action of the unitary group on the orbits.

The controllability on semisimple Lie groups introduced in control theory in [3] was formulated in terms of generically verified sufficient controllability conditions in e.g. [4, 5, 6, 7] using the structure theory of semisimple Lie algebras [1]. Some of these conditions for compact Lie groups, drawn from [7], will be reformulated here as sufficient conditions for the complex flag manifolds in a form which is still generic and very straightforward to verify.

These conditions can be generalized also to the case of multiple systems having drifts of different amplitude and driven by the same control vector field [8, 9]. In the spirit of this work, the Lie algebraic rank conditions of [8, 9] are rephrased in terms of sufficient conditions and proven to be generic also for this class of multiple systems: it is shown that simultaneous controllability holds always except for identical systems (up to sign).

A particular case of the situation under study originates from the control of finite dimensional non-dissipative bilinear quantum systems, see [10, 11] for the structure of the state space, [12, 13, 14, 15, 16, 17] for quantum control notions. In this case, the Hermitian (positive semidefinite) matrix represents a density operator and the isospectral evolution the so-called Liouville-von Neumann equation [18]. The relation between controllability of the wavefunction and of the density operator has already been studied in the literature [12, 16]. In particular the Lie algebraic rank condition was already used in [12]. Here we show that controllability is generic also for density operators having a spectrum which is nondegenerate and with nondegenerate transitions between energy levels.

The simultaneous controllability principle [9, 8] is used to show controllability for ensembles of quantum systems having different Larmor precession frequencies [19].

2 Model formulation

Consider the space $\mathcal{H}$ of $N \times N$ Hermitian matrices $\mathcal{H} = \{X = X^\dagger \;:\; X \in \mathbb{C}^{N\times N}\}$ and the matrix ODE

$$\dot{X} = [A + uB,\, X], \quad X \in \mathcal{H}, \tag{1}$$

with $u \in C^\infty(\mathbb{R})$ a control function, and $A, B \in \mathfrak{u}(N)$ the Lie algebra of skew-Hermitian matrices, $A = -A^\dagger$, $B = -B^\dagger$. If $\mathcal{U}(N)$ is the group of unitary matrices, the solution of (1) is

$$X(t) = U(t)X(0)U^\dagger(t), \tag{2}$$

where $U(t) = \overrightarrow{\exp}\left(\int_0^t (A + u(\tau)B)d\tau\right) \in \mathcal{U}(N)$ is a formal exponential solution of the following ODE problem

$$\dot{U} = (A + uB)U, \quad U(0) = I, \quad U \in \mathcal{U}(N), \tag{3}$$

and the action (2) is called a *conjugation action*. An evolution like (1) is *isospectral* (or of a Lax type), i.e., the eigenvalues of X, call them $\Phi(X) = \{\eta_1, \dots, \eta_N\}$, form a complete set of constants of motion for (1). The vector space $\mathcal{H}$ is foliated under the conjugation action into (compact and connected) leaves uniquely determined by $\Phi(X)$. If $X_o \in \mathcal{H}$, the leaf passing through it, call it $\mathcal{S}_{X_o}$, is its $\mathcal{U}(N)$ adjoint orbit: $\mathcal{S}_{X_o} = \{UX_oU^\dagger,\ U \in \mathcal{U}(N)\}$. Since this action is transitive, $\mathcal{S}_{X_o}$ is a homogeneous space:

$$\mathcal{S}_{X_o} = \mathcal{U}(N)/\left(\mathcal{U}(j_1) \times \ldots \times \mathcal{U}(j_\ell)\right), \tag{4}$$

where $j_1, \dots, j_\ell$, $j_1 + \ldots + j_\ell = N$, $1 \leqslant \ell \leqslant N$, are the multiplicities of the ℓ distinct eigenvalues of $\Phi(X_o)$. The numbers $j_1, \dots, j_\ell$ form a flag in N, and the $\mathcal{S}_X$, $X \in \mathcal{H}$, are called *complex flag manifolds.* Since ℓ and $j_1, \dots, j_\ell$ change the dimension of the isotropy subgroup $\mathcal{U}(j_1) \times \ldots \times \mathcal{U}(j_\ell)$, different orbits can have different dimensions. Calling $m = \dim(\mathcal{S}_X)$, then m varies between the two extremes

1. $m = 2N - 2$: for rank-one states X_1 such that $\Phi(X_1) = \{1, 0, \dots, 0\}$,

$$\begin{aligned} S_{X_1} &= \mathcal{U}(N)/(\mathcal{U}(N-1) \times \mathcal{U}(1)) \\ &= \mathcal{U}(N)/(\mathcal{U}(N-1) \times \mathbb{S}^1); \end{aligned}$$

2. $m = N^2 - N$: for X_2 with all different eigenvalues $\Phi(X_2) = \{\eta_1, \ldots, \eta_N\}$, $\eta_j \neq \eta_\ell$,

$$S_{X_2} = \mathcal{U}(N)/\mathcal{U}(1)^N = \mathcal{U}(N)/(\mathbb{S}^1)^N$$

which is the generic case.

For any ℓ, $j_1, \ldots, j_\ell$: $2N - 2 \leqslant m \leqslant N^2 - N$.

Concerning the Lie algebra $\mathfrak{u}(N)$, let $\mathfrak{h}$ be the Cartan subalgebra of $\mathfrak{u}(N)$, i.e., the maximally abelian subalgebra in $\mathfrak{u}(N)$, $\dim(\mathfrak{h}) = N$. Denote $\mathfrak{k}$ the vector space such that $\mathfrak{u}(N) = \mathfrak{h} \oplus \mathfrak{k}$, with $\mathfrak{h} \perp \mathfrak{k}$ in the biinvariant metric $\mathrm{tr}\left(AB^\dagger\right)$, A, $B \in \mathfrak{u}(N)$. A "natural" basis for $\mathfrak{u}(N)$ associates $\mathfrak{h}$ with the diagonal matrices and $\mathfrak{k}$ with the off-diagonal ones. See e.g. [1] for more details on the decomposition of semisimple Lie algebras and on the structure of the homogeneous spaces.

3 Controllability on complex flag manifolds

Given $X_o \in \mathcal{S}_X$, the rechable set of (1) from X_o is defined as $\mathcal{R}(X_o) = U_{0 \leqslant t \leqslant \infty} \mathcal{R}(X_o, t)$ where $\mathcal{R}(X_o, t) = \{X \in \mathcal{S}_X \text{ s. t. } X(0) = X_o \text{ and } X(t) = X,\, t > 0, \text{ for some admissible control } u : [0,\, t] \to \mathbb{R}\}$.

Definition 1 *The system* (1) *is controllable at* $X_o \in \mathcal{S}_X$ *if* $\mathcal{R}(X_o) = \mathcal{S}_X$.

The controllability of a system on a semisimple Lie group like (3) is a well-studied problem [3, 6, 20, 7]. In particular, we shall focus on the conditions provided in [7].

Assuming without loss of generality that $A \in \mathfrak{h}$, in the basis mentioned above A will be diagonal:

$$A = -i \begin{bmatrix} \mathcal{E}_1 & & \\ & \ddots & \\ & & \mathcal{E}_N \end{bmatrix},$$

with the $\mathcal{E}_j \in \mathbb{R}$ supposed ordered: $\mathcal{E}_1 \leqslant \mathcal{E}_2 \leqslant \ldots \leqslant \mathcal{E}_N$.

Definition 2 $A \in \mathfrak{u}(N)$ *is said* regular *if* $\mathcal{E}_i \neq \mathcal{E}_j$, $i \neq j$. *A regular* A *is said* strongly regular *if* $\mathcal{E}_i - \mathcal{E}_j \neq \mathcal{E}_p - \mathcal{E}_q$, $(i, j) \neq (p, q)$ $i \neq j$, $p \neq q$.

The set of strongly regular elements is open and dense in $\mathfrak{u}(N)$ [4]. Concerning the input matrix $B = [b_{ij}]$, we shall assume that $B \in \mathfrak{k}$ i.e., B is off-diagonal in the same basis as above. Consider the *graph* $\mathcal{G}_B$ associated to the matrix B, i.e., the pair $\mathcal{G}_B = (\mathcal{N}_B, \mathcal{C}_B)$

where $\mathcal{N}_B$ represents a set of N ordered nodes $\mathcal{N}_B = \{1, \dots, N\}$ and $\mathcal{C}_B$ the set of arcs joining the nodes: $\mathcal{C}_B = \{(i,\, j) \mid b_{ij} \neq 0\}$. Since B is skew-Hermitian, $\mathcal{G}_B$ is symmetric, hence nonoriented.

Definition 3 *The graph $\mathcal{G}_B$ is said* connected *if for all pairs of nodes in $\mathcal{N}_B$ there exists an oriented path in $\mathcal{C}_B$ connecting them.*

The following is a weaker definition of strong regularity [6].

Definition 4 *$A \in \mathfrak{u}(N)$ is said B-*strongly regular *if $\mathcal{E}_i - \mathcal{E}_j \neq \mathcal{E}_p - \mathcal{E}_q$, $(i, j) \neq (p, q)$ $i \neq j$, $p \neq q$ for all (i, j), $(p, q) \in \mathcal{C}_B$.*

Clearly A strongly regular is B-strongly regular for any B.

On a semisimple Lie algebra like $\mathfrak{u}(N)$, the accessibility condition $\mathrm{Lie}(A,\, B) = \mathfrak{u}(N)$ is generic, i.e., the set of pairs A, B that fulfill it is open and dense in $\mathfrak{u}(N)$, see [4] Thm. 12, Ch. 6. Since $\mathfrak{u}(N)$ is compact, this condition implies controllability on the Lie group $\mathcal{U}(N)$. The following theorem provides sufficient conditions for $\mathrm{Lie}(A,\, B) = \mathfrak{u}(N)$. It is proven in [7] (Theorem 3) for $\mathfrak{su}(N)$. The proof is identical for $\mathfrak{u}(N)$.

Theorem 1 *Assume A is B-strongly regular and that $\mathcal{G}_B$ is connected. Then* $\mathrm{Lie}(A,\, B) = \mathfrak{u}(N)$ *and the system* (3) *is controllable.*

Now let us turn to the isospectral evolution (1).

Theorem 2 *Assume A is B-strongly regular and that $\mathcal{G}_B$ is connected. Then the system* (1) *is controllable on each orbit $\mathcal{S}_X \subset \mathcal{H}$.*

Proof. Theorem 1 implies that for the system (3) $\mathcal{R}(I) = \mathcal{U}(N)$. Given $X(0) = X_o \in \mathcal{S}_X$, then it follows from the transitivity of the conjugation action on $\mathcal{S}_X$ that for (1) $\mathcal{R}(X_o) = \{U X_o U^\dagger \,:\, U \in \mathcal{U}(N)\} = \mathcal{S}_X$. □

Just like for $\mathcal{U}(N)$, the condition is generic and independent from the initial condition.

Corollary 1 *The system* (1) *is almost always controllable, i.e., it is controllable for almost all pairs A, $B \in \mathfrak{u}(N)$ and any initial condition $X_o \in \mathcal{S}_X$.*

4 Simultaneous controllability

Assume we have r different drift terms $A_1, \ldots, A_r \in \mathfrak{u}(N)$ and consider the r bilinear systems on the complex flag manifold (possibly on different leaves of the foliation):

$$\begin{aligned} \dot{X}_j &= [A_j + uB,\, X_j], \quad X_j \in \mathcal{S}_{X_{jo}}, \quad j = 1, \ldots, r \\ X_j(0) &= X_j. \end{aligned} \tag{5}$$

The simultaneous controllability problem [8, 9] consists in checking if $\exists\; u : [0, t] \to \mathbb{R}$ able to steer simultaneously the r systems (5) to any desired point of $\mathcal{S}_{X_1} \times \ldots \times \mathcal{S}_{X_r}$, i.e., if $\mathcal{R}(X_{1o} \times \ldots \times X_{ro}) = \mathcal{S}_{X_1} \times \ldots \times \mathcal{S}_{X_r}$. Denote $\alpha_j = \mathrm{tr}\,(A_j)$, $\beta = \mathrm{tr}\,(B)$, $\hat{A}_j = A_j - \alpha_j I_N$, $\hat{B} = B - \beta I_N$, and $d = \mathrm{rank} \begin{bmatrix} \alpha_1 & \beta \\ \vdots & \vdots \\ \alpha_r & \beta \end{bmatrix}$.

Theorem 3 *([8]) Consider the following block diagonal matrices,* $\bar{A} = diag\left(\hat{A}_1, \ldots, \hat{A}_r\right)$, $\bar{B} = diag\left(\hat{B}, \ldots, \hat{B}\right)$*. The r systems* (5) *are simultaneously controllable if*

$$dim\left(Lie\left(\bar{A},\, \bar{B}\right)\right) = r(N^2 - 1) + d.$$

Hence, in order to have simultaneous controllability one needs to have controllability on each simple ideal (determined by $\mathrm{Lie}(\hat{A}_j,\, \hat{B})$, i.e., on each subsystem), plus one needs also to exclude pathological cases in which $\mathrm{Lie}(\hat{A}_j,\, \hat{B}) = \mathfrak{su}(N)$ $\forall j = 1, \ldots, r$, but $\mathrm{Lie}(\bar{A}_j,\, \bar{B}) \subsetneq \underbrace{\mathfrak{su}(N) \oplus \ldots \oplus \mathfrak{su}(N)}_{r \text{ times}}$ ("$\oplus$" is a direct sum of vector spaces).

Generalizing the condition of [9] (valid for two-level quantum systems), we show next that these pathological cases correspond to subsystems that are identical up to sign.

Theorem 4 *Given* $\hat{A}_1, \ldots, \hat{A}_r, \hat{B} \in \mathfrak{su}(N)$, $\hat{A}_i = \epsilon_i \hat{A}_1$, $\epsilon_i \neq \pm 1$, $\epsilon_i \neq \epsilon_j$, $i, j = 2, \ldots, r$, *if* $\hat{A}_1$ *strongly regular and* $\mathcal{G}_B$ *connected, then the r systems* (5) *are simultaneously controllable.*

Proof. For the sake of simplicity we consider only the case $r = 2$. The extension to any r follows the same scheme. Under the assumptions of the theorem, $\mathrm{Lie}(\hat{A}_j,\, \hat{B}) = \mathfrak{su}(N)$. Clearly if $\hat{A}_2 = \hat{A}_1$, the commutators in the two diagonal blocks of

$$[\bar{A},\, \bar{B}] = \begin{bmatrix} [\hat{A}_1,\, \hat{B}] & 0 \\ 0 & [\hat{A}_2,\, \hat{B}] \end{bmatrix}$$

and of all higher order commutators are the same. As the two diagonal blocks are ideals and each sequence of Lie brackes that allows one system to completely generate $\mathfrak{su}(N)$ also works for the other ideal, then $\mathrm{Lie}(\bar{A},\ \bar{B}) = \mathrm{Lie}(\hat{A},\ \hat{B}) = \mathfrak{su}(N)$ in this case, i.e., simultaneous controllability is not achieved. If $\hat{A}_2 = -\hat{A}_1$, then

$$\begin{aligned} [\hat{A}_2,\ \hat{B}] &= -[\hat{A}_1,\ \hat{B}] \\ [\hat{A}_2,\ [\hat{A}_2,\ \hat{B}]] &= [\hat{A}_1,\ [\hat{A}_1,\ \hat{B}]] \end{aligned}$$

while

$$[[\hat{A}_2,\ \hat{B}],\ \hat{B}] = -[[\hat{A}_1,\ \hat{B}],\ \hat{B}]]$$

meaning that if a basis can be generated using commutators containing only an odd (or only an even) number of $\hat{A}_1$, then the Lie algebras have still indistinguishable generators as in the previous case. If instead $\hat{A}_2 = \epsilon\hat{A}_1$, $\epsilon \neq \pm 1$, then we can use the fact that $\hat{A}_1 \in \mathfrak{h}$, $\hat{B} \in \mathfrak{k}$, and the particular structure of the $[\mathfrak{h},\ \mathfrak{k}]$ commutators in $\mathfrak{su}(N)$. Recall (see e.g. [21]), that $\mathfrak{k}$ admits the splitting into root spaces $\mathfrak{k} = \bigoplus_{1\leqslant i<j\leqslant N} \mathfrak{k}_{ij}$ and that $\hat{B} = [b_{ij}]_{1\leqslant i<j\leqslant N}$ intersects the $\mathfrak{k}_{ij}$ subspaces whenever $b_{ij} \neq 0$. Assume the number of $b_{ij} \neq 0$ is m. Consider the subspace $\mathcal{W} = \mathrm{span}\left\{\hat{A}_1,\ \hat{B},\ \mathrm{ad}_{\hat{A}_1}\hat{B},\ \ldots,\mathrm{ad}^k_{\hat{A}_1}\hat{B}\right\}$. The conditions $\hat{A}_1$ strongly regular and $\mathcal{G}_B$ connected guarantee that $\mathrm{Lie}(\mathcal{W}) = \mathfrak{su}(N)$ (Theorem 2 of [7]), and in particular that $\mathcal{W}$ is spanned by $k = 2m-1$ commutations, i.e.,

$$\mathrm{span}\left\{\bar{A}_1,\ \bar{B},\ \mathrm{ad}_{\bar{A}_1}\bar{B},\ \ldots,\mathrm{ad}^{2m-1}_{\bar{A}_1}\bar{B}\right\} = \begin{bmatrix} \mathcal{W} & 0 \\ 0 & \mathcal{W} \end{bmatrix}.$$

If $\dim(\mathcal{W}) = 2m+1$, then $\exists$ a basis $E_1, \ldots, E_{2m+1}$ of $\mathcal{W}$ (each E_i expressed as a linear combination of the generating fields $\hat{A}_1,\ \hat{B},\ \mathrm{ad}_{\hat{A}_1}\hat{B},\ \ldots,\mathrm{ad}^{2m-1}_{\hat{A}_1}\hat{B}$) such that

$$\mathrm{ad}^k_{\hat{A}_1}\hat{B} = \mu_1 E_1 + \ldots + \mu_{2m+1}E_{2m+1}, \quad \forall k > 2m-1.$$

However, since $\epsilon \neq \pm 1$, $\epsilon^k\mathrm{ad}^k_{\hat{A}_1}\hat{B} \neq \mu_1 E_1 + \ldots + \mu_{2m+1}E_{2m+1}, \quad k > 2m-1$. Hence for the coupled system we can write the linear combination of commutators (all available at the k-th step, $k > 2m-1$):

$$\mathrm{ad}^k_{\bar{A}}\bar{B} - \mu_1\begin{bmatrix} E_1 & 0 \\ 0 & E_1 \end{bmatrix} - \ldots - \mu_{2m+1}\begin{bmatrix} E_{2m+1} & 0 \\ 0 & E_{2m+1} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & (\epsilon-1)^k\mathrm{ad}^k_{\hat{A}_1}\hat{B} \end{bmatrix} = \bar{C}$$

which is clearly linearly independent from $\mathrm{ad}_{\bar{A}}\bar{B}, \ldots, \mathrm{ad}_{\bar{A}}^{2m-1}\bar{B}$ (it only acts on the second system). From the commutation relations

$$[\mathfrak{h}, \mathfrak{k}_{ij}] = \mathfrak{k}_{ij},$$

we can deduce that $\mathrm{ad}_{\hat{A}_1}^{k}\hat{B}$ never vanishes for any order k and keeps generating the $\mathfrak{k}_{ij}$ subspaces. Hence $\mathrm{ad}_{\bar{A}}\bar{C}, \ldots, \mathrm{ad}_{\bar{A}}^{2m-1}\bar{C}$ are all linearly independent Lie brackets (all acting only on the second system). By combining these commutators with the previous ones, we have generated independently the two vector spaces

$$\bar{\mathcal{W}}_1 = \begin{bmatrix} \mathcal{W} & 0 \\ 0 & 0 \end{bmatrix}, \quad \text{and } \bar{\mathcal{W}}_2 = \begin{bmatrix} 0 & 0 \\ 0 & \mathcal{W} \end{bmatrix}.$$

As for each of them $\mathrm{Lie}(\bar{\mathcal{W}}_j) = \mathfrak{su}(N)$, the result follows. □

Remark 1 As in Theorem 2, the strong regularity of $\hat{A}_1$ in Theorem 4 can be relaxed to B-strong regularity.

Remark 2 As pointed out in [9], the pathological cases correspond to higher dimensional representations of a Lie algebra. If the systems are identical, all Lie brackets on the subsystems are identical and the Lie algebra generated is isomorphic to that of a single system.

Remark 3 The condition of Theorem 4 should probably extend to any nonidentical $\hat{A}_1, \ldots, \hat{A}_r$ not necessarily linearly dependent. However, finding an explicit set of generating commutators is a more involved task. The same problem is encountered in correspondence of systems with different control vector fields B.

Remark 4 Notice that $\mathcal{S}_{X_i}$ and $\mathcal{S}_{X_j}$ need not to be the same leaf of the flag foliation, i.e., systems on manifolds possibly of different dimensions can be steered simultaneously (all leaves are transitive under the same group action).

5 Application to quantum ensembles: Liouville-von Neumann equation

In quantum mechanics, the set of admissible density operators for an N-level quantum system is defined as an affine convex cone in $\mathcal{H}$:

$$\mathcal{M} = \{X = X^\dagger \geqslant 0,\ \mathrm{tr}(X) = 1,\ \mathrm{tr}(X^2) \leqslant 1\} \subset \mathcal{H}.$$

Hence the orbits $\mathcal{S}_X \in \mathcal{H}$ represent quantum density operators if and only if $\Phi(X) = \{\eta_1, \ldots, \eta_N\}$, $0 \leqslant \eta_j \leqslant 1$, $\sum_{j=1}^N \eta_j = 1$.

The density operator formalism, introduced by J. von Neumann [18], is meant to describe *statistical ensembles* of quantum systems: if (in Dirac notation) we consider the wavefunctions $|\psi\rangle_j$ of unit norm, $\langle\psi_j|\psi_j\rangle = 1$, $|\psi\rangle_j \in \mathbb{S}^{2N-1} \subset \mathbb{C}^N$, $j = 1, \ldots, N$, their convex combination

$$X = \sum_{j=1}^{k} p_j |\psi_j\rangle\langle\psi_j| \in \mathcal{M}, \qquad p_j \geqslant 0, \quad \sum_{j=1}^{k} p_j = 1$$

is called a density operator. See e.g. the standard textbooks [22, 18] for a thorough description, or [23] for a readable introduction for non-physicists.

The density operator X obeys to an isospectral ODE which is normally referred to as the Liouville-von Neumann equation [18]. Just like with each $|\psi_j\rangle$ one can consider a forced Schrödinger equation given by a free Hamiltonian H_A and a driving Hamiltonian H_B and study the corresponding control problem, see [7, 14, 24, 17], so denoting $H_A = iA$ and $H_B = iB$, one obtains that (1) can be thought of as a forced Liouville-von Neumann equation [13, 12, 16].

When $k = 1$ we have the so-called *pure states* for which $X_1 = |\psi\rangle\langle\psi|$ is a rank one matrix. In this case, the orbit $\mathcal{S}_{X_1}$ is the base manifold of the Hopf fibration $\mathbb{S}^{2N-1} \xrightarrow{\mathbb{S}^1} \mathcal{S}_{X_1} = \mathbb{C}P^{N-1}$, with fibers representing the global phase. All cases with $k \geqslant 2$ are referred to as *mixed states* and correspond to flag manifolds $\mathcal{S}_X$ of dimension $m > 2N - 2$.

From Theorem 2 of Section 3, it follows that the *same* sufficient conditions hold simultaneously for the Schrödinger and the Liouville-von Neumann equations, and that for both they are *generically* verified. In terms of the spectrum of $H_A = iA$, the strong regularity condition means nondegenerate energy levels $\mathcal{E}_j$ with nondegenerate transition frequencies $\mathcal{E}_i - \mathcal{E}_j$, while connectivity of the graph $\mathcal{G}_B$ means that the control vector field has to enable all energy transitions.

Corollary 2 *(of Theorem 2) A driven Liouville-von Neumann equation is almost always controllable in $\mathcal{S}_X$.*

Problems of simultaneous control of multiple systems are frequent in quantum control [8, 9, 25, 19], see also [26] for a related problem. We shall consider here the case of an ensemble of N-level systems having Larmor dispersion i.e., systems

$$\dot{X} = -i[\lambda H_A + uH_B, X], \quad \lambda \in [\lambda_o - \lambda_d, \lambda_o + \lambda_d], \ \ 0 < \lambda_d < \lambda_o. \tag{6}$$

Corollary 3 *(of Theorem 4) Any ensemble of systems in* (6) *with nonidentical Larmor frequencies $\lambda_1, \ldots, \lambda_r \in [\lambda_o - \lambda_d, \lambda_o + \lambda_d]$ is simultaneously controllable.*

This result generalizes the findings of [19] for ensembles of spin 1/2 systems. Extensions to the inhomogeneous field case are not considered here (but cold be treated by similar means).

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