Feedback stabilization of isospectral control systems on complex flag manifolds: application to quantum ensembles

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***Abstract*—The convex set of density operators of an N-level quantum mechanical system foliates as a complex flag manifold, where each leaf is identified with the adjoint unitary orbit of the eigenvalues of a density matrix. For an isospectral bilinear control system evolving on such an orbit, the state feedback stabilization problem admits a natural Lyapunov-based time-varying feedback design. A global description of the domain of attraction of the closed-loop system can be provided based on a "root-space"-like structure of the cone of density operators. The converging conditions are time-independent but depend on the topology of the flag manifold: it is shown that the closed loop must have a number of equilibria at least equal to the Euler characteristic of the manifold, thus imposing topological obstructions to global stabilizability.**

***Index Terms*—Feedback stabilization, Bilinear control systems, Quantum control, Convergence analysis.**

I. Introduction

The state feedback stabilization problem for bilinear control systems has been studied for a long time, see e.g. [24], [22]. The common setting adopted in all these works is always that of a state space which is $\mathbb{R}^n$, with the origin as equilibrium point to be stabilized. In this paper, instead, we focus on a particular class of bilinear (matrix) control systems, defined on a family of compact manifolds and evolving isospectrally. The original formulation comes from quantum control of non-dissipative systems [10], [14], [15], with the state matrix representing a quantum mechanical density operator and the isospectral evolution the so-called Liouville von-Neumann equation [39]. The problems connected with the phenomenon of wavefunction collapse following a measurement, (see [28], [30] for a thorough account of the peculiarities of quantum measurements or [37] for a control theoretic perspective), are bypassed by considering density operators of quantum ensembles and completely noninvasive measurements (i.e., classical measurements: with a back-action which is negligible in the limit of large ensembles). This also allows us to relax the requirement of commutativity of the measured observables and in fact we shall assume to have a complete knowledge of the density operator (the state) for all times. Hence, in control terms, we assume to be dealing with a classical state feedback stabilization problem. Although physically this set up is realistic only for some applications (typically nuclear spin ensembles [13], [23]), it is of widespread use for the purposes of model-based quantum state steering (often under the name "tracking control" [10], [41]), as it allows us to generate control functions also for difficult tasks in spite of the high complexity of the open loop control problem [8], [15], [34]. See [3] for an application to the dipolar decoupling problem of identical spin systems.

While the formulation comes from quantum control, the main motivations for this work are of a mathematical nature, namely feedback design and convergence analysis for a class of bilinear control systems defined on the so-called *complex flag manifolds* [6], [31], [42]. These are a family of compact manifolds foliating the convex set of $N \times N$ positive semidefinite Hermitian matrices of trace 1 (the density operators), that can be described as the orbits of the density operators under the $SU(N)$-conjugation action. Such evolution is isospectral, as the eigenvalues of the density operator form a complete set of invariants of an orbit, while their multiplicity determine its dimension [6], [11], [16].

Since the bilinear system has a drift term which cannot be canceled without introducing singularities in the control law, the most natural problem formulation is to seek for a stabilizer to the periodic trajectory drawn by the drift. Rather than studying this problem like an orbital stabilization problem [5], we reformulate and solve it as a state tracking problem, thus avoiding the obstruction to semiglobal convergence of a periodic orbit, see [40], Corollary 1.6 (where it is called stability in the large). In fact, with our feedback design the state will converge to a periodic trajectory evolving on the orbit of the drift. As a matter of fact, by passing to a suitable rotating frame, our time-dependent trajectory tracking problem can be reformulated completely in terms of time-varying feedback law for the fixed point of a nonautonomous system.

The Lyapunov design is essentially of the Jurdjevic-Quinn type [24], for which the usual LaSalle invariance principle is applicable in spite of the time-dependence of the closed loop, and does not differ much from what has already been proposed in the literature for wavefunctions [17], [38], [20], [27].

What is nontrivial is to ascertain the convergence of a given initial condition and to provide a global description of the region of attraction. In fact, the "global" sufficiency criterion used in [24] to prove asymptotic convergence and based on the so-called ad-commutators [4], is never verified for $N > 2$. For wavefunctions, a related condition based on the controllability of the linearization along the desired reference trajectory was shown in [27] to be a local sufficient condition for stabilizability. Both conditions fail to give a global convergence analysis because of the nontrivial topological structure of a complex flag manifold.

It is known [7], [26], that compact manifolds without boundary do not admit a global asymptotically stable equilibrium because they are not contractible. This is a topological property and corresponds to a set being homotopy equivalent to a point [21]. The region of convergence of an asymptotically stable attractor must be in such a homotopy class [7], [40]. For our complex flag manifolds, it will be shown that a fundamental topological invariant like the Euler characteristic, which has as meaning the number of nontrivial possible permutations of the eigenvalues of the density operator, [6], [11], [16], [42] corresponds to the number of *antipodal* points, i.e., of equilibria of the closed-loop system representing unavoidable obstructions to global stabilizability.

It will be shown, however, that the undesired critical points are not only unstable but also repulsive, meaning that convergence is guaranteed for all initial conditions outside the set of equilibria. To attain a complete and time-independent description of the critical set and thus of the domain of attraction, we make use of the "overlap", up to the imaginary unit, between the set of (Hermitian) density operators and the Lie algebra $\mathfrak{su}(N)$ (of traceless skew-Hermitian matrices), and of a few tools deriving from the root space decomposition of a semisimple Lie algebra, namely its orthogonal decomposition into Cartan subalgebra plus root spaces and the invariance properties of the root spaces under certain commutators (like the ad-commutators) [2]. This "graph-like" approach yields simple, time-independent characterizations of all converging initial conditions for a given reference trajectory and Hamiltonian. Also the Kalman controllability of the linearization admits an intrinsic formulation in these terms. The characterization we obtain gives us insight into the problem of choosing reference trajectories having a large domain of attraction.

II. Driven Liouville-von Neumann equation

For a general introduction to the formalism of quantum mechanics we refer the reader to standard textbooks like e.g. [30], [39]. See also [28] for a readable introduction for non-physicists. More control oriented material for quantum systems can be found e.g. in [2], [10], [14], [15], [36].

In the semiclassical approximation [14], with a given Hamiltonian $H_A + uH_B$, $-iH_A, -iH_B \in \mathfrak{su}(N)$, $u \in C^\infty(\mathbb{R})$ a control function, one can form a Schrödinger equation for the wavefunction $|\psi\rangle$ (in atomic units, $\hbar = 1$, and with unit norm $\langle\psi|\psi\rangle = 1$)

$$|\dot{\psi}\rangle = -i\left(H_A + u\, H_B\right)|\psi\rangle, \quad |\psi(t)\rangle \in \mathbb{S}^{2N-1} \subset \mathbb{C}^N, \tag{1}$$

or a Liouville-von Neumann equation [39] for the density operator ρ

$$\dot{\rho} = -i[H_A + u\, H_B,\, \rho], \qquad \rho \in \mathcal{M}, \tag{2}$$

where $\mathcal{M} = \{\rho = \rho^\dagger \geqslant 0,\ \mathrm{tr}(\rho) = 1\}$ is a convex subset of the vector space of $N \times N$ Hermitian matrices. Eq. (2) holds for a quantum ensemble, hence it is more general than (1). When ρ can be written as the outer product $\rho = |\psi\rangle\langle\psi|$, then it is called a pure state and it is characterized by $\mathrm{tr}\,(\rho^2) = 1$. A state which is not pure is called mixed and for it $\mathrm{tr}\,(\rho^2) < 1$.

A. Structure of the state space

The evolution (2) is isospectral, i.e., the eigenvalues of ρ, $\Phi(\rho) = \{\eta_1, \ldots, \eta_N\}$, are constants of motion of (2). The convex set of all admissible density operators $\mathcal{M}$ is foliated into (compact and connected) leaves uniquely determined by $\Phi(\rho)$. Call $\mathcal{S} \in \mathcal{M}$ one such leaf and consider $\rho_o \in \mathcal{S}$. If the geometric multiplicities of the eigenvalues $\Phi(\rho_o)$ are given by $j_1, \ldots, j_\ell$, $j_1 + \ldots + j_\ell = N$, $2 \leqslant \ell \leqslant N$, then $\mathcal{S}$ is defined as (see e.g. [1], [9], [42])

$$\mathcal{S} = U(N)/\left(U(j_1) \times \ldots \times U(j_\ell)\right),$$

$j_1 + \ldots + j_\ell = N$, $2 \leqslant \ell \leqslant N$. As $j_1, \ldots, j_\ell$ form a flag in N, the homogeneous spaces $\mathcal{S}$ are called *complex flag manifolds*. Clearly, the dimension of $\mathcal{S}$, call it m, depends on the number of distinct eigenvalues and on their multiplicities. For example, for a pure state $\Phi = \{1, 0, \ldots, 0\}$, $\mathcal{S} = U(N)/\left(U(N-1) \times \mathbb{S}^1\right)$ and $\dim(\mathcal{S}) = 2N - 2$. At the other extreme, if $\Phi = \{\eta_1, \eta_2, \ldots, \eta_N\}$, $\eta_j \neq \eta_\ell$, $\sum_{j=1}^N \eta_j = 1$, then $\mathcal{S} = U(N)/\left(\mathbb{S}^1\right)^N$ has dimension $N^2 - N$. Hence $2N - 2 \leqslant m \leqslant N^2 - N$, m even.

In (2), if $\rho(0) \in \mathcal{S}$ then $\rho(t) \in \mathcal{S}$ $\forall\, t \geqslant 0$, and the reachable set of the bilinear control system is at most $\mathcal{S}$.

Choose now a suitable basis in which ρ_o is diagonal. The orbit $\mathcal{S}$ is transverse to the set of diagonal density operators, and meets it in a number of disjoint points equal to the number of distinct permutations of the entries of ρ_o. Such number is equal to the cardinality of the Weyl group as well as to the Euler characteristic $\chi(\mathcal{S})$ of the orbit, see [16], [42] and Theorem E.2 of [18]. These points form the vertices of a polygon in the $N-1$-dimensional diagonal "eigenensemble", and are sometimes called Weyl chambers. Inspired by the $\mathbb{S}^2$ case (see Example 1 below), we shall call them *antipodal*. If $\rho_o = \mathrm{diag}\,(\eta_1, \ldots, \eta_N)$, $\sum_{j=1}^N \eta_j = 1$, $0 \leqslant \eta_j \leqslant 1$, then the $\chi(\mathcal{S}) - 1$ antipodal points are given by $\mathrm{diag}\left(\eta_{\sigma(1)}, \ldots, \eta_{\sigma(N)}\right)$ with $\sigma(1), \ldots, \sigma(N)$ a permutation of $1, \ldots, N$ such that $\mathrm{diag}\left(\eta_{\sigma(1)}, \ldots, \eta_{\sigma(N)}\right) \neq \rho_o$. Since the orbits $\mathcal{S}$ originate from a transitive action, any point in $\mathcal{S}$ has the same structure as ρ_o.

B. Properties of $\mathfrak{su}(N)$ and Gell-Mann basis

For later use, we need to recall a few standard properties of the Lie algebra of traceless skew-Hermitian matrices $\mathfrak{su}(N)$ and its relation with $\mathcal{M}$. The reader is referred to e.g. [12], [33] for more details. Let $\mathfrak{h}$ be the Cartan subalgebra of $\mathfrak{su}(N)$, i.e., the abelian subalgebra of maximal dimension in $\mathfrak{su}(N)$, $\dim(\mathfrak{h}) = N - 1$. Let $\mathfrak{k}$ be the vector space such that $\mathfrak{su}(N) = \mathfrak{h} \oplus \mathfrak{k}$, with $\mathfrak{h} \perp \mathfrak{k}$ in a standard biinvariant $\mathfrak{su}(N)$ metric: $\mathrm{tr}\left(A^\dagger B\right)$, $A,\, B \in \mathfrak{su}(N)$. Since Hermitian matrices are related to skew-Hermitian matrices by a multiplication by the imaginary unit, up to i we can have the same complete orthonormal set covering both $\mathcal{M}$ and $\mathfrak{su}(N)$. Let $\lambda_0 = \frac{1}{\sqrt{N}} I_N$ and call $\boldsymbol{\lambda}$ the (N^2-1)-dimensional vector of $N \times N$ Gell-Mann matrices [19]. Then $\mathrm{span}\{-i\boldsymbol{\lambda}\} = \mathfrak{su}(N)$. In correspondence with $\mathfrak{su}(N) = \mathfrak{h} \oplus \mathfrak{k}$, we have the decomposition of $\boldsymbol{\lambda}$ into $\boldsymbol{\lambda}_\mathfrak{h}$ and $\boldsymbol{\lambda}_\mathfrak{k}$ so that $\mathfrak{h} = \mathrm{span}\{-i\boldsymbol{\lambda}_\mathfrak{h}\}$ and $\mathfrak{k} = \mathrm{span}\{-i\boldsymbol{\lambda}_\mathfrak{k}\}$, with $\mathfrak{h}$ corresponding to traceless, purely imaginary diagonal matrices and $\mathfrak{k}$ to off-diagonal skew-Hermitian matrices. If $E_{j\ell}$ is the elementary $N \times N$ matrix having 1 in the $(j\ell)$ slot and 0 elsewhere, then the matrices $\boldsymbol{\lambda}$ are given by

$$\begin{aligned} &\{\lambda_{\mathfrak{h},j}, \quad 1 \leqslant j \leqslant N-1\} = \\ &\left\{(E_{11} + \ldots + E_{jj} - jE_{j+1,j+1})/\sqrt{j(j+1)}\right\} \end{aligned} \tag{3}$$

$1 \leqslant j \leqslant N-1$, for the diagonal part, and

$$\left\{\lambda^{\Re}_{\mathfrak{k},j\ell},\ 1 \leqslant j < \ell \leqslant N\right\} = \left\{(E_{j\ell} + E_{\ell j})/\sqrt{2}\right\} \tag{4}$$

$$\left\{\lambda^{\Im}_{\mathfrak{k},j\ell},\ 1 \leqslant j < \ell \leqslant N\right\} = \left\{i(-E_{j\ell} + E_{\ell j})/\sqrt{2}\right\} \tag{5}$$

$1 \leqslant j < \ell \leqslant N$, for the off-diagonal part. Calling $\mathfrak{k}_{j\ell} = \mathrm{span}\left\{-i\lambda^{\Re}_{\mathfrak{k},j\ell}, -i\lambda^{\Im}_{\mathfrak{k},j\ell}\right\}$, then we have the further splitting of $\mathfrak{k}$ into "root spaces"

$$\mathfrak{k} = \bigoplus_{1 \leqslant j < \ell \leqslant N} \mathfrak{k}_{j\ell} \tag{6}$$

with the following commutation relations (see for instance [2] for the details):

$$[\mathfrak{h},\, \mathfrak{k}_{j\ell}] = \mathfrak{k}_{j\ell}, \tag{7}$$

$$[\mathfrak{k}_{j\ell},\,\mathfrak{k}_{pq}] = \begin{cases} \emptyset & \text{if } \ell \neq p \text{ and } j \neq q \\ \mathfrak{k}_{jq} & \text{if } \ell = p \\ \mathfrak{k}_{p\ell} & \text{if } j = q \\ \subseteq \mathfrak{h} & \text{if } j = p \text{ and } \ell = q. \end{cases} \tag{8}$$

In terms of (3)-(5), any density ρ can be decomposed as $\rho = \varrho_0\lambda_0 + \rho_{\mathfrak{h}} + \rho_{\mathfrak{k}} = \varrho_0\lambda_0 + \sum_{1\leqslant j<N} \varrho_{\mathfrak{h},j}\lambda_{\mathfrak{h},j} + \sum_{1\leqslant j<\ell\leqslant N}\left(\varrho^{\Re}_{\mathfrak{k},j\ell}\lambda^{\Re}_{\mathfrak{k},j\ell} + \varrho^{\Im}_{\mathfrak{k},j\ell}\lambda^{\Im}_{\mathfrak{k},j\ell}\right)$, where $\varrho_0 = \frac{1}{\sqrt{N}}$, and $\varrho_{\mathfrak{h},j} = \mathrm{tr}(\rho\lambda_{\mathfrak{h},j}) \in \mathbb{R}$, $j = 1,\ldots,N-1$, $\varrho^{\Re}_{\mathfrak{k},j\ell} = \mathrm{tr}(\rho\lambda^{\Re}_{\mathfrak{k},j\ell}) \in \mathbb{R}$, $\varrho^{\Im}_{\mathfrak{k},j\ell} = \mathrm{tr}(\rho\lambda^{\Im}_{\mathfrak{k},j\ell}) \in \mathbb{R}$, $1 \leqslant j < \ell \leqslant N$ are called the expectation values of ρ along the basis elements [28].

Denote $\mathfrak{f}_{\mathfrak{k}}(\rho)$ the "support" of ρ in $\mathfrak{k}$, i.e., the set of root spaces "touched" by ρ: $\mathfrak{f}_{\mathfrak{k}}(\rho) = \rho \cap \mathfrak{k}$. Also let $\mathcal{F}_{\mathfrak{k}}(\rho) = \{(j\ell) \text{ s.t. } \mathrm{tr}\left(\rho\mathfrak{k}_{j\ell}\right) \neq 0,\ 1 \leqslant j < \ell \leqslant N\}$ be the corresponding set of index pairs. When $(j\ell) \in \mathcal{F}_{\mathfrak{k}}(\rho)$, then $(\varrho^{\Re}_{\mathfrak{k},j\ell}, \varrho^{\Im}_{\mathfrak{k},j\ell}) \neq (0,0)$. Likewise $\mathfrak{f}_{\mathfrak{h}}(\rho) = \rho \cap \mathfrak{h}$, $\mathcal{F}_{\mathfrak{h}}(\rho) = \{(j) \text{ s.t. } \varrho_{\mathfrak{h},j} \neq 0,\ 1 \leqslant j < N\}$, and $\mathcal{F}(\rho) = \mathcal{F}_{\mathfrak{k}}(\rho) \cup \mathcal{F}_{\mathfrak{h}}(\rho)$.

Finally, if $C \in \mathfrak{su}(N)$ is decomposed in terms of the basis above as $C = -i\sum_{1\leqslant j<N} c_{\mathfrak{h},j}\lambda_{\mathfrak{h},j} - i\sum_{1\leqslant j<\ell\leqslant N}\left(c^{\Re}_{\mathfrak{k},j\ell}\lambda^{\Re}_{\mathfrak{k},j\ell} + c^{\Im}_{\mathfrak{k},j\ell}\lambda^{\Im}_{\mathfrak{k},j\ell}\right)$, we shall also indicate with $\mathfrak{f}_{\mathfrak{h}}(C)$, $\mathfrak{f}_{\mathfrak{k}}(C)$ the support of C in, respectively, $\mathfrak{h}$, $\mathfrak{k}$, of indexes $\mathcal{F}_{\mathfrak{h}}(C)$, $\mathcal{F}_{\mathfrak{k}}(C)$.

C. Properties of the Hamiltonian

We will make the following assumptions on the Hamiltonian of (2).

A1: H_A is diagonal and traceless

$$H_A = \begin{bmatrix} \mathcal{E}_1 & & \\ & \ddots & \\ & & \mathcal{E}_N \end{bmatrix}, \quad \mathcal{E}_1 + \ldots + \mathcal{E}_N = 0,$$

with the $\mathcal{E}_j$ supposed ordered:

$$\mathcal{E}_1 \leqslant \mathcal{E}_2 \leqslant \ldots \leqslant \mathcal{E}_N;$$

A2: H_A is *strongly regular*, i.e.,

1) the eigenvalues of H_A are nondegenerate: $\mathcal{E}_j \neq \mathcal{E}_\ell$, $j \neq \ell$;
2) the transition frequencies are nondegenerate: $\mathcal{E}_j - \mathcal{E}_\ell \neq \mathcal{E}_p - \mathcal{E}_q$, $(j\ell) \neq (pq)$ $j \neq \ell$, $p \neq q$.

A3: H_B is off-diagonal;

A4: H_B enables all transitions between adjacent eigenvalues: $\mathrm{tr}\left(H_B\mathfrak{k}_{j,j+1}\right) \neq 0\ \forall\ j = 1,\ldots,N-1$.

Beside connectivity of $\mathrm{Graph}(H_B)$ (i.e., the N-node graph having a link between the nodes j and ℓ when $(H_B)_{j\ell} \neq 0$), **A4** guarantees that all "fundamental root spaces" (see [2]) are excited by the dynamics.

D. Unforced equation

Proposition 1 *Assume* **A1÷A4** *hold and consider the system* (2). *The state ρ is an equilibrium point of* (2) *for $u = 0$ if and only if it is diagonal. Furthermore, if $\rho_{\mathfrak{k}} \neq 0$, then for $u = 0$*

1) $\mathfrak{f}_{\mathfrak{k}}(\rho(0)) = \mathfrak{f}_{\mathfrak{k}}(\rho(t))$;
2) *for δt small, $\varrho^{\Re}_{\mathfrak{k},j\ell}(t) \neq \varrho^{\Re}_{\mathfrak{k},j\ell}(t+\delta t)$ and $\varrho^{\Im}_{\mathfrak{k},j\ell}(t) \neq \varrho^{\Im}_{\mathfrak{k},j\ell}(t+\delta t)$ $\forall\ (j\ell) \in \mathcal{F}_{\mathfrak{k}}(\rho)$.*

Proof: When $u = 0$, for a given $\rho = \varrho_0\lambda_0 + \rho_{\mathfrak{h}} + \rho_{\mathfrak{k}}$,

$$[-iH_A,\,\rho_{\mathfrak{h}}] = 0, \tag{9}$$

from which it is obvious that ρ diagonal is a fixed point. To show the other direction, calling $\rho_{\mathfrak{k},j\ell} = \rho^{\Re}_{\mathfrak{k},j\ell} - i\rho^{\Im}_{\mathfrak{k},j\ell}$, then for the off-diagonal part of ρ

$$\rho_{\mathfrak{k},j\ell}(t) = e^{-i(\mathcal{E}_j - \mathcal{E}_\ell)t}\rho_{\mathfrak{k},j\ell}(0). \tag{10}$$

A1÷A2 imply that $\mathcal{E}_j - \mathcal{E}_\ell \neq 0\ \forall\ (j\ell)$, $1 \leqslant j < \ell \leqslant N-1$, hence from (10) whenever $\rho_{\mathfrak{k},j\ell}(0) \neq 0$, $\rho_{\mathfrak{k},j\ell}(t) \neq 0\ \forall\, t > 0$. Condition 1 and 2 of the last part also follows straightforwardly from (10). □

III. Feedback stabilization for N-level quantum ensembles

A. Problem formulation

For the system (2), we are interested in the problem of tracking the periodic orbit drawn by the free Hamiltonian H_A. More precisely, the stabilization problem is the following.

Given the system (2) *with $\rho \in \mathcal{S}$, find $u = u(\rho_d(t), \rho)$ such that, for $t \to \infty$, $\rho(t) \to \rho_d(t)$, where $\rho_d(t)$ obeys*

$$\dot{\rho}_d(t) = [-iH_A,\,\rho_d(t)], \tag{11}$$

$-iH_A \in \mathfrak{su}(N)$, $\rho_d(t) \in \mathcal{S}$.

This is a full state tracking problem which, from Proposition 1, reduces to stabilization to an equilibrium point when $\rho_d(t)$ is diagonal. In the following ρ_d will always mean the reference trajectory $\rho_d(t)$ solution of (11).

B. Modified Jurjevic-Quinn conditions and antipodal points

The algorithm for the feedback design resembles the one used for $|\psi\rangle$ in [17], [38], [20], [27] and indeed the standard Jurdjevic-Quinn method for bilinear systems [24]. It consists in choosing as candidate Lyapunov function the Frobenius norm $V = V(\rho_d(t), \rho) = \mathrm{tr}\,(\rho_d^2(t)) - \mathrm{tr}\,(\rho_d(t)\rho)$. If $\rho_d(t)$ obeys (11),

$$\begin{aligned}\dot V =&\dot V(\rho_d(t), \rho) = -\mathrm{tr}\,([-iH_A,\, \rho_d(t)]\rho) \\ &- \mathrm{tr}\,([-iH_A,\, \rho]\rho_d(t)) - u\mathrm{tr}\,([-iH_B,\, \rho]\rho_d(t)) \\ =&u\mathrm{tr}\,([-iH_B,\, \rho_d(t)]\rho)\,.\end{aligned} \tag{12}$$

Since $\dot V$ is homogeneous in u, the obvious choice of feedback

$$u = -\mathrm{tr}\,([-iH_B,\, \rho_d(t)]\rho) \tag{13}$$

guarantees $\dot V = -(\mathrm{tr}\,([-iH_B,\, \rho_d(t)]\rho))^2 \leqslant 0$.

The following two sufficient conditions for asymptotic stability of the closed-loop system given by the feedback law (13) are adaptations of known results to the case at hand.

Proposition 2 *(Jurdjevic-Quinn [24]) Assume* **A1÷A4** *hold. Call* $\mathcal{W}^\ell = \mathrm{span}\left\{-iH_B,\, \mathrm{ad}_{-iH_A}(-iH_B), \ldots, \mathrm{ad}^\ell_{-iH_A}(-iH_B)\right\}$, *where* $\mathrm{ad}^\ell_{-iH_A}(-iH_B) = \underbrace{[-iH_A, \ldots, [-iH_A}_{\ell \text{ times}} -iH_B]]$. *If* $\mathcal{W}^\ell = \mathfrak{su}(N)$ *for some* ℓ*, then the system* (2) *with the feedback* (13) *is asymptotically converging to* $\rho_d(t)$ *from all* $\rho(0) \in \mathcal{S}$ *which are not antipodal points of* $\rho_d(0)$.

Proposition 3 *(Mirrahimi-Rouchon [27]) Assume* **A1÷A4** *hold and call* $\mathcal{Q}^\ell(\rho_d(t)) = \mathrm{span}\left\{[-iH_B,\, \rho_d(t)],\, [-iH_A,\, [-iH_B, \rho_d(t)]], \ldots, \mathrm{ad}^\ell_{-iH_A}[-iH_B, \rho_d(t)]\right\}$. *If the condition*

$$\dim \mathcal{Q}^{m-1}(\rho_d(t)) = m \tag{14}$$

is satisfied $\forall\, \rho_d(t) \in \mathcal{S}$ *obeying* (11)*, then the system* (2) *with the feedback* (13) *is locally asymptotically converging to* $\rho_d(t)$.

Both criteria are based on the application of LaSalle invariance principle to the closed loop system. Concerning Proposition 2, the largest invariant set $\mathcal{E}$ in $\mathcal{N} = \{\rho \in \mathcal{S} \text{ s.t. } \dot V(\rho_d(t), \rho) = 0\}$ can be computed looking at the locus in which $u = \frac{du}{dt} = \ldots = \frac{d^\ell u}{dt^\ell} = 0$ in correspondence with the feedback law (13):

$$\frac{du}{dt} = -\mathrm{tr}\,([\,[-iH_A,\, -iH_B],\, \rho_d(t)\,]\,\rho) = 0$$

and, similarly,

$$\frac{d^\ell u}{dt^\ell} = (-1)^\ell \mathrm{tr}([\,\mathrm{ad}^\ell_{-iH_A}(-iH_B),\, \rho_d(t)]\rho) = 0. \tag{15}$$

Hence, if $\mathcal{W}^\ell = \mathfrak{su}(N)$ for some ℓ, $\mathcal{E}$ contains no other trajectory than $\rho_d(t)$ and convergence follows.

For $|\psi\rangle$ which is an eigenfunction, the condition of Proposition 3, used implicitly in [38], is made explicit in [27] as a Kalman rank condition on the linear tangent system. Given $\rho \in \mathcal{S}$, the linearization of (2) around $\rho_d(t)$ yields

$$\frac{d\rho}{dt} = [-iH_A,\, \rho(t)] + u[-iH_B,\, \rho_d(t)] \in T_{\rho_d}\mathcal{S}. \tag{16}$$

Since $\dim(T_{\rho_d}\mathcal{S}) = m$, if (16) satisfies condition (14), then in the same spirit of the original Jurdjevic-Quinn work (see Lemma 2 below for the details), this implies that in closed loop $\mathcal{E}$ contains no other trajectory than $\rho_d(t)$, at least locally.

These criteria are not straightforwardly useful for at least a couple of reasons:

1) the Lie algebraic condition of Proposition 2 never applies (see Lemma 1);
2) the linearized tangent system may be time-varying and the sufficient condition is always local.

Both arguments are consequences of the nontrivial topology of $\mathcal{S}$, and imply that the domain of attraction of the closed-loop system cannot be the entire $\mathcal{S}$.

The vector space of Proposition 2, $\mathcal{W}^\ell$, is invariant under the so-called "ad-brackets" [24] but not necessarily a Lie subalgebra. For the system (2) we have that these ad-brackets are never generating. Call $\mathcal{W}^\ell_A = \mathrm{span}\{-iH_A,\, \mathcal{W}^\ell\}$.

Lemma 1 *Under* **A1÷A4***, for the system* (2), $\mathrm{Lie}(\mathcal{W}^\ell_A) = \mathfrak{su}(N)$ *but* $\mathcal{W}^\ell_A \subsetneq \mathfrak{su}(N)\ \forall\, \ell > 0$ *if* $N \geqslant 3$*, while* $\mathcal{W}^\ell \subsetneq \mathfrak{su}(N)\ \forall\, \ell > 0$ *if* $N \geqslant 2$.

Proof: Since H_A is strongly regular and **A4** implies that $\mathrm{Graph}(H_B)$ is connected, it follows from Theorem 2 of [2] that the smallest subalgebra containing $-iH_A$, $-iH_B$ is $\mathfrak{su}(N)$. For the second part, recall that $\dim(\mathfrak{h}) = N - 1$. From the Lie bracket relations (7), $-iH_A \in \mathfrak{h}$, $-iH_B \in \mathfrak{k}$ implies $\mathrm{ad}^{\ell}_{-iH_A}(-iH_B) \in \mathfrak{k}$. Hence $\mathcal{W}^{\ell} \subset \mathfrak{k}\ \forall\, N \geqslant 2$. Even adding $-iH_A$, $\mathcal{W}^{\ell}_A$ alone cannot fully generate $\mathfrak{h}$ for any ℓ if $N \geqslant 3$. □

The first part of Lemma 1 is also known as the strong accessibility condition [29]. Since $\mathfrak{su}(N)$ is compact, it suffices for controllability on each orbit $\mathcal{S}$.

Concerning Proposition 3, it is easy to see that even when $\rho_d(t)$ is diagonal and (16) is time-invariant, the Kalman-like "rank condition" (14) alone is not enough to guarantee attractivity, as can be checked for example in any other diagonal state $\rho_p \in \mathcal{S}$: in fact $[-iH_A, \rho_p] = 0$ i.e., the linear system turns out to be driftless. This argument can be generalized as follows.

Proposition 4 *Consider the system* (2) *with the feedback* (13)*. Given* $\rho_d(t) \in \mathcal{S}$ *obeying* (11)*, any antipodal state* $\rho_p(0) \in \mathcal{S}$ *of* $\rho_d(0)$ *is such that* $\rho_p(t)$ *remains antipodal to* $\rho_d(t)\ \forall\, t \geqslant 0$*.*

Proof: For any given diagonal pair $\rho_p(0)$, $\rho_d(0)$, $[-iH_B, \rho_p(0)] \in \mathfrak{k}/i$ is off-diagonal. But then $u = -\mathrm{tr}\left([-iH_B,\, \rho_d(0)]\rho_p(0)\right) = 0$, since $\mathfrak{h} \perp \mathfrak{k}$. Hence no feedback is produced and since $\rho_d(t) = \rho_d(0)$, $\rho_p(t) = \rho_p(0)\ \forall\, t > 0$, the claim follows. Consider now the case of $\rho_d(0),\, \rho_p(0) \in \mathcal{S}$ antipodal but not diagonal. For the unforced system, one has that $\rho_d(0)$, $\rho_p(0)$ antipodal implies $\rho_d(t)$, $\rho_p(t)$ antipodal for all $t > 0$. To see this, notice that by the transitivity of the $SU(N)$ action on $\mathcal{S}$, $\exists\ U_1 \in SU(N)$ such that $\tilde{\rho}_d(0) = U_1\rho_d(0)U_1^{\dagger}$ and $\tilde{\rho}_p(0) = U_1\rho_p(0)U_1^{\dagger}$ are both diagonal. But then $\rho_d(t) = e^{-iH_At}\rho_d(0)e^{iH_At} = e^{-iH_At}U_1^{\dagger}\tilde{\rho}_d(0)U_1e^{iH_At}$, and similarly $\rho_p(t) = e^{-iH_At}U_1^{\dagger}\tilde{\rho}_p(0)U_1e^{iH_At}$, i.e., $\rho_d(t)$ and $\rho_p(t)$ are still diagonalizable by the same unitary matrix and therefore still antipodal. For $\rho_d(0)$ and $\rho_p(0)$ antipodal, the feedback (13) at $t = 0$ is $u(0) = -\mathrm{tr}\left([-iH_B,\, \rho_d(0)]\rho_p(0)\right) = -\mathrm{tr}\left(-iH_BU_1^{\dagger}[\tilde{\rho}_d(0),\, \tilde{\rho}_p(0)]U_1\right) = 0$. Since no feedback is produced, the system remains unforced and, for what said above, $u(t) = 0\ \forall\, t \geqslant 0$.

□

As will be shown in next Section, the antipodal points are not the only states lacking attractivity, and the linearization alone is not enough to investigate the domain of attraction of the feedback stabilizer.

In the case of $\rho_d(t)$ diagonal, the antipodal points are equilibria for the closed loop system. For $\rho_d(t)$ nondiagonal, they are critical periodic trajectories.

Remark 1 The trajectory tracking problem presented above admits a reformulation as a point stabilization for a nonautonomous system. Consider a frame rotating with $-iH_A$. Call $\widehat{\rho}_d$ and $\widehat{\rho}$ the new reference and state matrices. Then $\widehat{\rho}(t) = e^{iH_At}\rho(t)e^{-iH_At}$ and $\widehat{\rho}_d(t) = e^{iH_At}\rho_d(t)e^{-iH_At} = \rho_d(0)$, i.e., the reference trajectory becomes a *fixed point*. Using a variation of constants formula, we obtain for (2)

$$\begin{cases} \dot{\widehat{\rho}}(t) &= u\,[e^{iH_At}(-iH_B)e^{-iH_At},\, \widehat{\rho}(t)] \\ \widehat{\rho}(0) &= \rho(0). \end{cases} \tag{17}$$

The Lyapunov function $V = \mathrm{tr}\left(\widehat{\rho}_d^{\,2}\right) - \mathrm{tr}\left(\widehat{\rho}_d\widehat{\rho}\right)$ and its derivative $\dot{V} = u\mathrm{tr}\left([-iH_B,\, \widehat{\rho}_d]\widehat{\rho}\right)$ are invariant to the change of reference frame. The uniformity in time of the asymptotic stability for the nonautonomous system (17) with the same feedback stabilizer as (13) follows directly.

C. Time-independent convergence conditions

The following Theorem provides a time-independent condition for asymptotic stabilizability to any $\rho_d(t) \in \mathcal{S}$, and a global description of the region of attraction of the controller.

Theorem 1 *Assume* **A1÷A4** *hold and consider the system* (2) *with the feedback* (13)*, where* $\rho_d(t) \in \mathcal{S}$ *obeys* (11)*. An initial condition* $\rho(0) \in \mathcal{S}$ *is asymptotically converging to* $\rho_d(t)$ *if*

1) $\rho(0)$ *is not an antipodal point of* $\rho_d(0)$*,*
2) $\mathcal{F}\left([-iH_B,\, \rho_d(t)]\right) \cap \mathcal{F}\left(\rho(0)\right) \neq 0$*,*
3) $\mathrm{Card}\mathcal{F}_{\mathfrak{k}}\left([-iH_B,\, \rho_d(t)]\right) \geqslant m/2$*,*

where Card *denotes the number of pairs of indexes in* $\mathcal{F}_{\mathfrak{k}}$*.*

In order to prove the Theorem we need a few preliminary results. The following Lemma implies that although the vector space $\mathcal{W}^{m-1}$ is never the entire Lie algebra $\mathfrak{su}(N)$ acting transitively on $\mathcal{S}$, it may nevertheless span the entire tangent space at a point. The same holds for the Kalman controllability. An equivalent time-independent condition is then provided.

Lemma 2 *Under the assumptions* **A1÷A4***, the following three conditions are equivalent:*

1) *the Kalman-like condition* (14) *is satisfied;*
2) $\dim([\mathcal{W}^{m-1}, \rho_d(t)]) = m$*;*
3) $\mathrm{Card}\mathcal{F}_{\mathfrak{k}}([-iH_B, \rho_d(t)]) \geqslant m/2$.

Proof: Given $C \in \mathfrak{su}(N)$, strong regularity of H_A implies that $C, [-iH_A, C], \ldots, \mathrm{ad}^{\ell-1}_{-iH_A} C$ are all linearly independent up to $\ell = 2\,\mathrm{Card}\mathcal{F}_{\mathfrak{k}}(C)$, see Theorem 2 in [2]. If $C = C_{\mathfrak{h}} + C_{\mathfrak{k}}$, since $[-iH_A, C_{\mathfrak{h}}] = 0$, only the off-diagonal part of C matters. The support $\mathfrak{f}_{\mathfrak{k}}(C)$ intersects a number of "root spaces" $\mathfrak{k}_{j\ell}$ (each has real dimension 2) equal to $\mathrm{Card}\mathcal{F}_{\mathfrak{k}}(C)$. Furthermore, since $-iH_A \in \mathfrak{su}(N)$, from (7) and **A2** it follows that:

$$\mathfrak{f}_{\mathfrak{k}}\left(\mathrm{ad}^{\ell}_{-iH_A} C\right) = \mathfrak{f}_{\mathfrak{k}}(C), \tag{18}$$

while

$$\mathfrak{f}_{\mathfrak{h}}\left(\mathrm{ad}^{\ell}_{-iH_A} C\right) = 0. \tag{19}$$

If $C = [-iH_B, \rho_d(t)]$ as in (16), then we have the Kalman-like controllability condition (14) provided that $\mathrm{Card}\mathcal{F}_{\mathfrak{k}}([H_B, \rho_d(t)]) \geqslant m/2$, and (18) implies

$$\mathcal{F}_{\mathfrak{k}}([-iH_B, \rho_d(t)]) = \mathcal{F}_{\mathfrak{k}}(\mathrm{ad}^{\ell}_{-iH_A}[-iH_B, \rho_d(t)]), \tag{20}$$

$\ell = 1, 2, \ldots$. If instead $C = -iH_B$, one gets the ℓ-th order commutator of $\mathcal{W}^{m-1}$. From the Jacobi identity it follows that for any $\ell > 0$

$$\begin{aligned}
&[\mathrm{ad}^{\ell}_{-iH_A}(-iH_B), \rho_d(t)] = \\
&\quad = [[-iH_A, \mathrm{ad}^{\ell-1}_{-iH_A}(-iH_B)], \rho_d(t)] \\
&\quad = [-iH_A, [\mathrm{ad}^{\ell-1}_{-iH_A}(-iH_B), \rho_d(t)]] \\
&\qquad - [\mathrm{ad}^{\ell-1}_{-iH_A}(-iH_B), [-iH_A, \rho_d(t)]]
\end{aligned} \tag{21}$$

and, by induction on ℓ,

$$[\mathcal{W}^{\ell}, \rho_d(t)] = \mathcal{Q}(\rho_d(t)). \tag{22}$$

In summary, strong regularity of H_A guarantees the full spanning of a linear space whose dimension is determined uniquely by $\mathrm{Card}\mathcal{F}_{\mathfrak{k}}([-iH_B, \rho_d(t)])$. The equivalence of the three conditions follows consequently. $\square$

Remark 2 In general $\mathrm{Card}\mathcal{F}_{\mathfrak{k}}(H_B) \neq \mathrm{Card}\mathcal{F}_{\mathfrak{k}}([-iH_B, \rho_d(t)])$, hence the controllability of the linearization depends from the reference trajectory $\rho_d(t)$ chosen.

Remark 3 While the Kalman-like condition (14) seems time-varying as soon as $\rho_{d,\mathfrak{k}}(t) \neq 0$, the equivalent condition 3 of Lemma 2 is always time-independent since $\mathcal{F}_{\mathfrak{k}}([-iH_B, \rho_d(0)]) = \mathcal{F}_{\mathfrak{k}}([-iH_B, \rho_d(t)])$.

Remark 4 The conditions of Lemma 2 depend on $\rho_d(t)$, H_A and H_B but not on the state ρ, meaning that alone they are not enough to guarantee convergence of a given $\rho(0)$.

The Lyapunov derivative in (12) is made homogeneous in u by the cancellation of the drift term and therefore the notion of attractivity provided by $\dot{V}$ must be rendered invariant under such flow (in a way similarly to the orbital stabilization problem, see [5]). The following Lemma gives an alternative attractivity condition which is fully invariant under the drift and generically equivalent to the usual Lyapunov convergence property. This last in fact may fail in isolated points: certain critical points of V are not invariant under the flow of the drift (see Section IV for examples).

Lemma 3 *Assume* **A1÷A4** *hold, and consider the system* (2) *with the feedback* (13)*, where* $\rho_d(t)$ *obeys to* (11)*. The following conditions are generically equivalent under the flow of the drift term:*
1) $\mathcal{F}([-iH_B, \rho_d(t)]) \cap \mathcal{F}(\rho) \neq 0$*;*
2) $\dot{V}(\rho_d(t), \rho) < 0$*;*
3) $\mathrm{tr}([\mathrm{ad}^{\ell}_{-iH_A}(-iH_B), \rho_d(t)]\,\rho) \neq 0 \;\forall\, \ell > 0$.

Proof: Clearly $\dot{V} = -(\mathrm{tr}([-iH_B, \rho_d(t)]\rho))^2 < 0$ implies $\mathcal{F}([-iH_B, \rho_d(t)]) \cap \mathcal{F}(\rho) \neq 0$. To prove that also the converse is generically true, it is enough to show that when $\mathcal{F}([-iH_B, \rho_d(t)]) \cap \mathcal{F}(\rho) \neq 0$ the zero crossing of the inner product can occur only at isolated points along the trajectories of the closed loop system. Assume Condition 1 holds and, at time t, $\mathrm{tr}([-iH_B, \rho_d(t)]\rho) = 0$. If δt is a small time increment, then from Condition 1 of Proposition 1, $\mathcal{F}([-iH_B, \rho_d(t)])$ and $\mathcal{F}(\rho)$ remain the same, while, from Condition 2 of Proposition 1

$$\begin{aligned}
(\rho^{\Re}_{\mathfrak{k},j\ell}(t+\delta t), \rho^{\Im}_{\mathfrak{k},j\ell}(t+\delta t)) &\neq (\rho^{\Re}_{\mathfrak{k},j\ell}(t), \rho^{\Im}_{\mathfrak{k},j\ell}(t)) \\
\rho_{\mathfrak{h},j}(t+\delta t) &= \rho_{\mathfrak{h},j}(t) \\
(\rho^{\Re}_{d,\mathfrak{k},j\ell}(t+\delta t), \rho^{\Im}_{d,\mathfrak{k},j\ell}(t+\delta t)) &\neq (\rho^{\Re}_{d,\mathfrak{k},j\ell}(t), \rho^{\Im}_{d,\mathfrak{k},j\ell}(t)) \\
\rho_{d,\mathfrak{h},j}(t+\delta t) &= \rho_{d,\mathfrak{h},j}(t).
\end{aligned}$$

If $\mathcal{F}_{\mathfrak{h}}([-iH_B, \rho_d(t)]) \cap \mathcal{F}_{\mathfrak{h}}(\rho) \neq 0$, then from the last row of (8) only $\rho_{d,\mathfrak{k}}(t)$ matters in the computation of $\mathcal{F}_{\mathfrak{h}}([-iH_B, \rho_d(t)])$, and $\mathrm{tr}([-iH_B, \rho_{d,\mathfrak{k}}(t+\delta t)]\rho_{\mathfrak{h}}(t+\delta t)) \neq 0$ since $\rho_{d,\mathfrak{k}}(t+\delta t) \neq \rho_{d,\mathfrak{k}}(t)$, while $\rho_{\mathfrak{h}}(t+\delta t) = \rho_{\mathfrak{h}}(t)$. If, instead, $\mathcal{F}_{\mathfrak{k}}([-iH_B, \rho_d(t)]) \cap \mathcal{F}_{\mathfrak{k}}(\rho) \neq 0$, then we have two possible contributions to consider:

$\mathcal{F}_{\mathfrak{k}}([-iH_B,\, \rho_{d,\mathfrak{h}}(t)])$ and $\mathcal{F}_{\mathfrak{k}}([-iH_B,\, \rho_{d,\mathfrak{k}}(t)])$. In the first case the conclusion follows from the same argument used above since now $\rho_{d,\mathfrak{h}}(t+\delta t) = \rho_{d,\mathfrak{h}}(t)$ while $\rho_{\mathfrak{k}}(t+\delta t) \neq \rho_{\mathfrak{k}}(t)$. In the second case it follows from the observation that $\mathcal{F}_{\mathfrak{k}}([-iH_B,\, \rho_{d,\mathfrak{k}}(t)]) \cap \mathcal{F}_{\mathfrak{k}}(\rho) \neq 0$ implies $\mathcal{F}_{\mathfrak{k}}(\rho_{d,\mathfrak{k}}(t)) \neq \mathcal{F}_{\mathfrak{k}}(\rho)$ (see the explicit computations of the commutators in (27)). The general case $\mathcal{F}([-iH_B,\, \rho_d(t)]) \cap \mathcal{F}(\rho) \neq 0$ is the sum of the two situations just described. Concerning Item 3, it is enough to notice that generically $\mathrm{tr}([-iH_B,\, \rho_d(t)]\rho) \neq 0$ if and only if $\mathrm{tr}\left([\mathrm{ad}^{\ell}_{-iH_A}(-iH_B),\, \rho_d(t)]\rho\right) \neq 0$. The argument is of the same type used in the proof of Lemma 2. For example if $\mathcal{F}_{\mathfrak{k}}([-iH_B,\, \rho_{d,\mathfrak{k}}(t)]) \cap \mathcal{F}_{\mathfrak{k}}(\rho) \neq 0$ then just apply

$$\mathcal{F}_{\mathfrak{k}}([-iH_B, \rho_d(t)]) = \mathcal{F}_{\mathfrak{k}}([\mathrm{ad}^{\ell}_{-iH_A}(-iH_B), \rho_d(t)]),$$

$\ell = 1,\, 2, \ldots$. The genericity of the argument can be shown as above. □

Proof: (of Theorem 1) For the closed loop system, consider the set $\mathcal{N}$ of critical points. Condition 3 guarantees that locally around $\rho_d(t)$ there is no other closed loop trajectory in $\mathcal{N}$, as, from Lemma 2, the linearization (16) at $\rho_d(t)$ is controllable. Hence $\rho_d(t)$ is a locally asymptotically stable (time-varying) equilibrium for the closed loop system and $\rho_d(t)$ is isolated in $\mathcal{N}$. Consider $\rho_e(t) \in \mathcal{N}$, $\rho_e(t) \neq \rho_d(t)$, $\rho_e(t)$ obeying (11). This implies $\rho_e(t)$ disjoint from $\rho_d(t)$ and $V(\rho_d(t), \rho_e(t)) > 0$. The independent variable t will be omitted from now on. We need to show that ρ_e must be a repulsive critical trajectory for the closed loop system [1]. For any $\rho_e \in \mathcal{N}$, it is enough to perturb ρ_e to $\tilde{\rho}_e \in \mathcal{S}$ so that $\mathcal{F}([-iH_B,\, \rho_d]) \cap \mathcal{F}(\tilde{\rho}_e) \neq 0$. It is always possible to do this in a neighborhood of ρ_e since $\mathcal{F}([-iH_B,\, \rho_d])$ has cardinality at least $m/2$ and $(h^{\Re}_{B,j\,j+1}, h^{\Im}_{B,j\,j+1}) \neq (0,\, 0)$ implies that $\mathrm{Graph}(H_B)$ is connected and that there is no subspace $\mathfrak{k}_{j\ell}$ invariant under $-iH_B$. But then, from Lemma 3, $\mathrm{tr}([-iH_B, \rho_d]\tilde{\rho}_e) \neq 0$ and $\dot{V}(\rho_d, \tilde{\rho}_e) < 0$, i.e., $\tilde{\rho}_e$ is attracted to ρ_d. To show that $V(\rho_e, \tilde{\rho}_e)$ increases, assume by contradiction that

$$\begin{aligned}\dot{V}(\rho_e, \tilde{\rho}_e) &= -\mathrm{tr}(\dot{\rho}_e\tilde{\rho}_e) - \mathrm{tr}\left(\rho_e\dot{\tilde{\rho}}_e\right) \\ &= -\mathrm{tr}([-iH_B, \tilde{\rho}_e]\rho_d)\ \mathrm{tr}([-iH_B, \tilde{\rho}_e]\rho_e) < 0. \end{aligned} \tag{23}$$

[1] Since ρ_e may not be isolated in $\mathcal{N}$, the term repulsive has to be intended as “semi-repulsive”.

Consider the geodesic line in $\mathcal{S}$ of reference trajectories connecting ρ_d with ρ_e: $\rho_\phi(s) = \rho_d + \phi(s)$ such that $\phi(0) = 0$ and $\phi(s_e) = \rho_e - \rho_d$ and $\rho_\phi(s)$ obeying (11) for all $s \in [0,\, s_e]$. Along this line,

$$\begin{aligned}\dot{V}(\rho_\phi(s), \tilde{\rho}_e) = &-\mathrm{tr}([-iH_B, \tilde{\rho}_e]\rho_d)^2 \\ &- \mathrm{tr}([-iH_B, \tilde{\rho}_e]\rho_d)\ \mathrm{tr}([-iH_B, \tilde{\rho}_e]\phi(s)),\end{aligned}$$

$s \in [0, s_e]$, is a function linear in $\phi(s)$ and such that, by the assumption (23),

$$\begin{aligned}\dot{V}(\rho_\phi(0), \tilde{\rho}_e) &= \dot{V}(\rho_d, \tilde{\rho}_e) = -\mathrm{tr}([-iH_B, \tilde{\rho}_e]\rho_d)^2 < 0 \\ \dot{V}(\rho_\phi(s_e), \tilde{\rho}_e) &= \dot{V}(\rho_e, \tilde{\rho}_e) < 0.\end{aligned}$$

But then $\dot{V}(\rho_\phi(s), \tilde{\rho}_e) < 0\ \forall\ s \in [0, s_e]$ and $\dot{V}(\rho_\phi(s), \rho_\phi(s)) = 0$. Since $\tilde{\rho}_e$ is any point in $\mathcal{S} \setminus \mathcal{N}$ and $V(\cdot\,, \cdot)$ is a distance on $\mathcal{S}$ such that $V(\rho_\phi(s), \tilde{\rho}_e) > 0$ and $V(\rho_\phi(s), \rho_\phi(s)) = 0$, it follows that each $\rho_\phi(s)$ $s \in [0, s_e]$ is a (time-varying) equilibrium for the closed loop system which is at least locally stable. But this is a contradiction, since ρ_d is an isolated locally asymptotically stable (time-varying) equilibrium. Hence it must be that $\dot{V}(\rho_e, \tilde{\rho}_e) \geqslant 0$ i.e., ρ_e must be repulsive. Therefore ρ_e cannot belong to the invariant set $\mathcal{E}$. From Lemma 3, all conditions (15) are satisfied or violated simultaneously when $\dot{V} = 0$ or $\dot{V} < 0$ respectively, i.e., when $\mathcal{F}([-iH_B,\, \rho_d]) \cap \mathcal{F}(\tilde{\rho}_e) = 0$ or $\neq 0$. Hence outside $\mathcal{N}$ $\rho(0)$ must converge to ρ_d since any other $\rho_e \in \mathcal{N}$ is repulsive. □

Remark 5 Condition 2 of Theorem 1 is obviously a necessary condition for convergence. Condition 3 instead is sufficient but not necessary, see Example 1 in Section IV.

While, from Lemma 2, the linear spans at $\rho_d(t)$ of the linearized system (16) and of the $\mathcal{W}^{\ell}$ yield a space of the same dimension, Condition 3 of Lemma 3 holds for each of the bilinear forms (in $\rho_d(t)$ and ρ) but it is in general not true for the linearization at $\rho_d(t)$.

Corollary 1 *Under* **A1÷A4** *and for* $\rho_d(t)$ *obeying* (11)*:*

1) $\dim\left([\mathcal{W}^{\ell}, \rho_d(t)]\right) = \dim \mathcal{Q}^{\ell}(\rho_d(t))$, $\forall\, \ell = 0, \ldots, m-1$;
2) $\mathrm{tr}([\mathrm{ad}^{\ell}_{-iH_A}(-iH_B),\, \rho_d(t)]\rho) \neq 0$ $\not\Leftrightarrow \mathrm{tr}((\mathrm{ad}^{\ell}_{-iH_A}[-iH_B, \rho_d(t)])\rho) \neq 0$.

Proof: The first condition is a consequence of the strong regularity of H_A and follows straightforwardly from Lemma 2, see (22). The second claim follows from the recursive application of the Jacobi identity (21) and the observation that $\mathrm{ad}^{\ell}_{-iH_A} C_{\mathfrak{h}} = 0$, $C \in \mathfrak{su}(N)$. □

The consequence is that the linearization alone is inconclusive about the region of attraction of the reference trajectory in the closed loop system, while instead the ad-commutators completely specify it.

Corollary 2 *When* $\mathrm{Card}\mathcal{F}_{\mathfrak{k}}([-iH_B,\, \rho_d(t)]) \geqslant m/2$, *the region of attraction of the system* (2) *with the feedback* (13) *is given by* $\mathcal{R} = \mathcal{S} \setminus \mathcal{N}$.

D. Global stabilization and topological obstructions

For a given control system, once a Lyapunov function and a feedback law are chosen, the cardinality and structure of the set of equilibria in closed loop can be investigated. Global feedback stabilization is achievable only if this set reduces to a single isolated point (or a time-varying reference trajectory as in our tracking formulation). From Proposition 4, for our choice of V and $u(\rho)$ there are at least as many equilibria as there are antipodal points on $\mathcal{S}$.

It would be interesting to draw the conclusion that this must always be the case for $\mathcal{S}$ regardless of the choice of V and $u = u(\rho)$, and that it is a consequence of the topological structure of $\mathcal{S}$. A manifold like $\mathcal{S}$, compact without boundary, cannot be globally asymptotically stabilized because it lacks the contractivity property, i.e., it is not homotopy equivalent to a point, see [7], Proposition 1 and Theorem 1, and [40]. This fact alone, however, sheds no light on the minimal number of equilibria of a feedback design.

Proposition 5 *Consider the system* (2) *and the (possibly time-varying) equilibrium* $\rho_d(t)$ *obeying* (11). *For any* $V \in C^\infty(\mathcal{S})$, $V(\rho) > 0$, $V(\rho_d(t)) = 0$ *and any smooth* $u = u(\rho)$ *such that for the closed-loop system* $\dot{V}(\rho) \leqslant 0$, $\dot{V}(\rho_d(t)) = 0$, *the set* $\mathcal{N} = \{\rho \in \mathcal{S}$ *s. t.* $\dot{V}(\rho) = 0\}$, *must contain at least* $\chi(\mathcal{S}) - 1$ *(possibly time-varying) equilibria.*

In order to prove the Proposition, we need the following result, straightforward adaptation of the main Theorem of [32].

Proposition 6 *Denote by* $\rho_{p_j}(t)$, $j = 1, \ldots, \chi(\mathcal{S}) - 1$, *the (possibly time-varying) antipodal points of* $\rho_d(t) \in \mathcal{S}$, *where* $\rho_d(t)$ *obeys* (11). *For each pair* $\rho_d(t)$, $\rho_{p_j}(t)$ *there exists a two-sphere* $\mathbb{S}^2_{\{\rho_d, \rho_{p_j}\}} \subset \mathcal{S}$ *such that* $\rho_d(t)$ *and* $\rho_{p_j}(t)$ *are antipodal points of* $\mathbb{S}^2_{\{\rho_d, \rho_{p_j}\}}$.

Proof: (of Proposition 5) From Proposition 6, $\mathcal{S}$ contains $\chi(\mathcal{S}) - 1$ two-spheres. Each two-sphere is itself a compact manifold without boundary, hence it is noncontractible. In particular, for any Lyapunov function $V \in C^\infty(\mathcal{S})$ denote V_j its restriction to $\mathbb{S}^2_{\{\rho_d, \rho_{p_j}\}}$. Then certainly for any choice of feedback $u = u(\rho)$ in closed loop the set $\mathcal{N}_j = \{\rho \in \mathbb{S}^2_{\{\rho_d, \rho_{p_j}\}}$ s. t. $\dot{V}_j(\rho) = 0\}$ contains at least two equilibria. This follows from $\dot{V}_j(\rho_d) = 0$ and $\dot{V}_j(\rho) \leqslant 0$ $\forall\, \rho \in \mathbb{S}^2_{\{\rho_d, \rho_{p_j}\}}$ (inherited from $\dot{V}(\rho) \leqslant 0\ \forall\, \rho \in \mathcal{S}$). In fact, if $\mathrm{Card}\,\mathcal{N}_j = 1$ then $\rho_d(t)$ would be globally stable in $\mathbb{S}^2_{\{\rho_d, \rho_{p_j}\}}$ noncontractible, which is a contradiction. By counting the number of distinct equilibria obtained in this way, $\mathrm{Card}\,\mathcal{N} \geqslant \chi(\mathcal{S})$ and the claim follows. □

IV. A few cases of physical interest

The methods developed above yield considerable insight into the stabilizability and convergence properties of a quantum density operator. A few interesting cases for N-level systems are now described. They are followed by a more detailed description for systems with $N = 2, 3$.

- Since $m \leqslant N^2 - N$, and $\mathrm{Card}\mathcal{F}_{\mathfrak{k}}([-iH_B, \rho_d(t)]) \leqslant (N^2 - N)/2$, (i.e., the maximal number of off-diagonal terms), each complex flag manifold $\mathcal{S}$ may admit a controllable linearization (depending on $\rho_d(t)$).
- The assumption **A4** of direct coupling between nearest energy levels is needed in order to exclude the existence of subsets of $\mathcal{S}$ which remain invariant under the closed loop dynamics. It is a common assumption in most practical cases (dipole approximation [14]). See also Example 2 below (last item).
- The full connectivity of $\mathrm{Graph}(H_B)$, i.e., $(H_B)_{ij} \neq 0\, \forall\, i \neq j$, is neither a sufficient nor a necessary condition for asymptotic stability.
- If ρ_d is an eigenstate and ρ another eigenstate then there is never convergence, not even if $\mathrm{Graph}(H_B)$ is fully connected, because ρ is antipodal to ρ_d.

- For pure states and a not fully connected $\mathrm{Graph}(H_B)$, certain eigenstates are easier to stabilize than others, since they have a larger region of attraction. The easiest is the one of energy $\mathcal{E}_j$ such that the index j appears more often in $\mathcal{F}_{\mathfrak{k}}\left([-iH_B, \rho_d]\right)$. In Example 2 below with H_B as in (24), the eigenstate of intermediate energy has a larger region of attraction than the ground state or the most excited state. When $\mathrm{Graph}(H_B)$ is fully connected, there is no such difference. From Theorem 1, this does not mean that all initial conditions have the same convergence properties to a given $\rho_d(t)$.
- If $\rho_d(t)$ and $\rho(0)$ are both block diagonal and the blocks do not overlap

$$\rho_d = \begin{bmatrix} \begin{array}{ccc|c} * & \dots & * & \\ \vdots & & \vdots & \\ * & \dots & * & \\ \hline & & & \\ & & & \\ & & & \end{array} \end{bmatrix},$$

$$\rho(0) = \begin{bmatrix} \begin{array}{c|ccc} & & & \\ & & & \\ \hline & * & \dots & * \\ & \vdots & & \vdots \\ & * & \dots & * \end{array} \end{bmatrix},$$

then

$$[-iH_B,\, \rho_d(t)] = \begin{bmatrix} \begin{array}{ccc|ccc} * & \dots & * & * & \dots & * \\ \vdots & & & & & \vdots \\ * & \dots & * & * & \dots & * \\ \hline * & \dots & * & 0 & \dots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ * & \dots & * & 0 & \dots & 0 \end{array} \end{bmatrix},$$

which implies $\mathcal{F}\left([-iH_B,\, \rho_d(t)]\right) \cap \mathcal{F}\left(\rho(0)\right) = 0$ and $\dot{V} = 0$, i.e., $\rho(0)$ is not attracted to $\rho_d(t)$.
- Not all states in $\mathcal{N}$ are maximally distant from $\rho_d(t)$. Assume ρ_d, ρ are such that $\mathcal{F}_{\mathfrak{h}}\left([-iH_B,\, \rho_d]\right) \cap \mathcal{F}_{\mathfrak{h}}\left(\rho(0)\right) \neq 0$, $\mathcal{F}_{\mathfrak{k}}\left(\rho_d\right) = 0$, $\mathcal{F}_{\mathfrak{k}}\left(H_B\right) \cap \mathcal{F}_{\mathfrak{k}}\left(\rho\right) = 0$. Also in this case $\mathcal{F}\left([-iH_B,\, \rho_d]\right) \cap \mathcal{F}\left(\rho(0)\right) = 0$ and ρ is not converging. However, since $\mathrm{tr}\left(\rho_d \rho\right) \neq 0$, ρ_d and ρ are not maximally distant.
- A typical example of an initial condition for which $\dot{V}(\rho_d(0), \rho(0)) = 0$ but not invariant under the drift (see paragraph before Lemma 3) is attained when $\mathcal{F}_{\mathfrak{k}}\left([-iH_B,\, \rho_d(t)]\right) \cap \mathcal{F}_{\mathfrak{k}}\left(\rho(0)\right) \neq 0$ but $[-iH_B,\, \rho_d(0)]$, $\rho(0)$ both real or purely imaginary. This follows from Proposition 1.

Example 1 $N = 2$. Consider ρ which is a pure state, $\Phi(\rho) = \{1, 0\}$. Topologically, the case $N = 2$ is the only easy one, as $\mathcal{S} = \mathbb{S}^2 \simeq \mathbb{C}P^1$. With the choice of basis (3)-(5), one has that $\rho_{\mathfrak{k}}$ lies on the great horizontal circle of $\mathbb{S}^2$, the diagonal antipodal states at the north and south poles, and $\mathfrak{h}$, $\dim(\mathfrak{h}) = 1$, corresponds to the vertical line passing through the poles of the sphere. Everything extends unchanged to mixed states ρ such that $\Phi(\rho) = \{\eta_1,\, \eta_2\}$, $\eta_1 + \eta_2 = 1$, $0 < \eta_1,\, \eta_2 < 1$, since $\mathcal{S}$ is still equal to $\mathbb{S}^2$. Since each $\mathcal{S}$ crosses $\mathfrak{h}$ exactly twice, $\chi(\mathcal{S}) = 2$.

Assume $H_A = \frac{h_{A,\mathfrak{h},1}}{\sqrt{2}} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = h_{A,\mathfrak{h},1} \lambda_{\mathfrak{h},1}$ and $H_B = \frac{h^{\Re}_{B,\mathfrak{k},12}}{\sqrt{2}} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = h^{\Re}_{B,\mathfrak{k},12} \lambda^{\Re}_{\mathfrak{k},12}$. From Proposition 1, both $\mathrm{tr}\left(\rho_{\mathfrak{k}}^2\right)$ and $\rho_{\mathfrak{h}}$ are integrals of motion of the unforced dynamics, while in coordinates the two components of $\rho_{\mathfrak{k}}$ evolve according to a sinusoidal law. In this case $\mathcal{W}^1 = \mathrm{span}\{-i\lambda^{\Re}_{\mathfrak{k},12},\, -i\lambda^{\Im}_{\mathfrak{k},12}\}$ and $\mathcal{W}^1_A = \mathfrak{su}(2)$. Hence the "global" condition of Proposition 2 is not valid. When applying Theorem 1 to the system plus the feedback (13), we have the following for the closed loop system:

- any $\rho_d(t)$ has a single antipodal point which is also an equilibrium;
- if $\rho_d(t)$ is diagonal, $\mathcal{F}_{\mathfrak{k}}\left([-iH_B, \rho_d]\right) = \{(12)\}$, the linearization is controllable and any non-diagonal ρ satisfies Theorem 1. Hence any $\rho(0)$ such that $\rho_{\mathfrak{k}}(0) \neq 0$ is in the domain of attraction of ρ_d diagonal;
- if $\rho_d(t)$ is off-diagonal, $\mathrm{Card}\mathcal{F}_{\mathfrak{k}}\left([-iH_B, \rho_d(t)]\right) = 0$, and the sufficient condition of Theorem 1 does not apply. However, $\mathcal{F}_{\mathfrak{h}}\left([-iH_B, \rho_d(t)]\right) \neq 0$ and as long as $\mathcal{F}_{\mathfrak{h}}\left([-iH_B, \rho_d(t)]\right) \cap \mathcal{F}_{\mathfrak{h}}\left(\rho(0)\right) \neq 0$, i.e., whenever $\rho_{\mathfrak{h}} \neq 0$, $\rho \to \rho_d$. This is a special situation due to $\dim(\mathfrak{h}) = 1$, and has no counterpart for $N > 2$.

In summary, there is always almost global convergence except when $\rho_{d,\mathfrak{h}} = \rho_{\mathfrak{h}} = 0$, i.e., except when both $\rho_d(t)$ and ρ belong to great horizontal circles. □

Example 2 $N = 3$, $\Phi(\rho) = \{1, 0, 0\}$ (pure state).

Since the isotropy subgroup in this case is $U(2) \times \mathbb{S}^1$ of dimension 5 (recall that $\dim(U(3)) = 9$), $\dim(\mathcal{S}) = 4$ and $\chi(\mathcal{S}) = 3$. The structure of $\mathcal{S} \subset \mathbb{S}^7$ in coordinates is studied in detail in [16], [11], [25], [35]. In particular, the single antipodal point of the case $N = 2$ is replaced by two symmetrically distributed and equidistant antipodal points.

For the all different eigenvalue case $\Phi(\rho) = \{\eta_1, \eta_2, \eta_3\}$, $\eta_j \neq \eta_k$, which is the generic case, the stabilizer is the torus $(\mathbb{S}^1)^3$, $\mathcal{S} = U(3)/(\mathbb{S}^1)^3$ and $\dim(\mathcal{S}) = 6$. The only diagonal matrices that are conjugate with $\rho_0 \in \mathcal{S}$ are its five element permutations, i.e., $\chi(\mathcal{S}) = 6$ in this case. The drift of the system is given by

$$\begin{aligned} H_A =& \frac{h_{A,\mathfrak{h},1}}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \frac{h_{A,\mathfrak{h},2}}{\sqrt{6}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \\ =& h_{A,\mathfrak{h},1}\lambda_{\mathfrak{h},1} + h_{A,\mathfrak{h},2}\lambda_{\mathfrak{h},2}. \end{aligned}$$

We shall consider the following control vector field

$$\begin{aligned} H_B =& \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & h^{\Re}_{B,\mathfrak{k},12} & 0 \\ h^{\Re}_{B,\mathfrak{k},12} & 0 & h^{\Re}_{B,\mathfrak{k},23} \\ 0 & h^{\Re}_{B,\mathfrak{k},23} & 0 \end{bmatrix} \\ =& h^{\Re}_{B,\mathfrak{k},12}\lambda^{\Re}_{\mathfrak{k},12} + h^{\Re}_{B,\mathfrak{k},23}\lambda^{\Re}_{\mathfrak{k},23}, \end{aligned} \tag{24}$$

which has $\mathcal{F}_{\mathfrak{k}}(H_B) = \{(12), (23)\}$ or, alternatively,

$$\begin{aligned} H_B =& \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & h^{\Re}_{B,\mathfrak{k},12} & h^{\Re}_{B,\mathfrak{k},13} \\ h^{\Re}_{B,\mathfrak{k},12} & 0 & h^{\Re}_{B,\mathfrak{k},23} \\ h^{\Re}_{B,\mathfrak{k},13} & h^{\Re}_{B,\mathfrak{k},23} & 0 \end{bmatrix} \\ =& h^{\Re}_{B,\mathfrak{k},12}\lambda^{\Re}_{\mathfrak{k},12} + h^{\Re}_{B,\mathfrak{k},13}\lambda^{\Re}_{\mathfrak{k},13} + h^{\Re}_{B,\mathfrak{k},23}\lambda^{\Re}_{\mathfrak{k},23} \end{aligned} \tag{25}$$

which has a "fully connected" graph, $\mathcal{F}_{\mathfrak{k}}(H_B) = \{(12), (13), (23)\}$.

A list of interesting cases is the following:

- any of the (two for pure, five for the generic case) antipodal points of any $\rho_d(t)$ is also an equilibrium.
- if $\rho_d(t)$ is diagonal only the off-diagonal part of ρ matters
  - if ρ_d is pure, e.g. $\rho_d = \mathrm{diag}(1, 0, 0)$
    - with H_B given in (24): $\mathcal{F}_{\mathfrak{k}}([-iH_B, \rho_d]) = \{(12)\} \Longrightarrow$ the linearization is never controllable since $2\,\mathrm{Card}\mathcal{F}_{\mathfrak{k}}([-iH_B, \rho_d]) < 4 = m$, hence Theorem 1 does not apply. Unlike the $N = 2$ case, now in general $\rho(0) \not\to \rho_d$;
    - with H_B given in (25): $\mathcal{F}_{\mathfrak{k}}([-iH_B, \rho_d]) = \{(12), (13)\} \Longrightarrow$ the linearization is controllable. Any $\rho(0)$ such that $\mathcal{F}_{\mathfrak{k}}(\rho(0)) \cap \{(12), (13)\} \neq 0$ is converging. However, if one considers the pure state

      $$\rho(0) = \frac{1}{2} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix},$$

      then $\mathcal{F}_{\mathfrak{k}}(\rho) = \{(23)\}$, implying $\dot{V}(0) = u = \mathrm{tr}([-iH_B, \rho_d(0)]\rho(0)) = 0$, i.e., the system is not converging to ρ_d in spite of the Kalman controllability condition on the linearization. Notice how for this example $\dim([\mathcal{W}^3, \rho_d]) = \dim([\mathcal{W}^3, \rho_d(0)]) = 4$, while $\mathrm{tr}([\mathrm{ad}^{\ell}_{-iH_A}(-iH_B), \rho_d]\rho) = 0$, $\ell = 0, 1, 2, 3$.
  - if ρ_d is pure, but $\rho_d = \mathrm{diag}(0, 1, 0)$
    - H_B is either (24) or (25): $\mathcal{F}_{\mathfrak{k}}([-iH_B, \rho_d]) = \{(12), (23)\} \Longrightarrow$ the linearization is always controllable. Any $\rho(0)$ such that $\mathcal{F}_{\mathfrak{k}}(\rho(0)) \cap \{(12), (23)\} \neq 0$ is converging.
  - if ρ_d has all different eigenvalues
    - for H_B as in (24): $\mathcal{F}_{\mathfrak{k}}([-iH_B, \rho_d]) = \{(12), (23)\} \Longrightarrow$ the linearization is never controllable since now $m = 6$;
    - for H_B as in (25): $\mathcal{F}_{\mathfrak{k}}([-iH_B, \rho_d]) = \{(12), (13), (23)\} \Longrightarrow$ the linearization is always controllable. Any $\rho(0)$ such that $\mathcal{F}_{\mathfrak{k}}(\rho(0)) \neq 0$ is converging; any $\rho(0)$ such that $\mathcal{F}_{\mathfrak{k}}(\rho(0)) = 0$ is antipodal.
- if $\rho_d(t) - \varrho_0\lambda_0$ is off-diagonal
  - for H_B as in (24) and $\mathcal{F}_{\mathfrak{k}}(\rho_d(t)) \subseteq \mathcal{F}_{\mathfrak{k}}(H_B) \Longrightarrow$ linearization is never controllable, hence Theorem 1 does not apply and in general $\rho(0) \not\to \rho_d(t)$;
  - for H_B as in (24) and $\mathcal{F}_{\mathfrak{k}}(\rho_d(t)) \not\subseteq \mathcal{F}_{\mathfrak{k}}(H_B)$ $\Longrightarrow \mathrm{Card}\mathcal{F}_{\mathfrak{k}}([-iH_B, \rho_d(t)])$ is at least 2, implying that the linearization is controllable at least for pure states;
  - if $\mathcal{F}_{\mathfrak{k}}(\rho_d(t)) \cap \mathcal{F}_{\mathfrak{k}}(H_B) \neq 0$, then also $\mathcal{F}_{\mathfrak{h}}(\rho(0))$ matters for the convergence, see (26);
  - if $\mathcal{F}_{\mathfrak{k}}(\rho_d(t)) \cap \mathcal{F}_{\mathfrak{k}}(H_B) = 0$, then convergence depends only on $\mathcal{F}_{\mathfrak{k}}(\rho(0))$ (plus controllability), see (27).

- If the control Hamiltonian is $H_B = h^{\Re}_{\mathfrak{k},12}\lambda^{\Re}_{\mathfrak{k},12} + h^{\Re}_{\mathfrak{k},13}\lambda^{\Re}_{\mathfrak{k},13}$, i.e., direct coupling between $\mathcal{E}_2$ and $\mathcal{E}_3$ is missing, then the sufficient condition of Theorem 1 does not apply. Assume for example

$$\rho_d(t) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{bmatrix}, \qquad \rho(0) = \begin{bmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Then $\mathcal{F}_{\mathfrak{k}}([-iH_B, \rho_d(t)]) = \{(12),(13)\}$ and $\mathcal{F}_{\mathfrak{k}}([-iH_B, \rho_d(t)]) \cap \mathcal{F}_{\mathfrak{k}}(\rho(0)) = \{(12)\}$. However, $\rho(0) \not\to \rho_d(t)$.

□

V. Conclusion

For a nonlinear system, attaining a global description of the region of attraction of a feedback control design is usually a very hard problem, especially when the manifold has a nontrivial topological structure and "competing" equilibria. Remarkably, the system studied in this work enjoys two properties that render a global description feasible: the set of critical points can be described exactly and the spurious equilibria are all repulsive.

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Appendix I
A few commutators

The following commutators from (7)-(8) are needed in the proof of Lemma 3 and in Example 2.

$$[\lambda_{\mathfrak{k},\Re,j\ell},\, \lambda_{\mathfrak{k},\Im,j\ell}] = i(E_{jj} - E_{\ell\ell}) = \begin{cases} -\sqrt{\frac{j-1}{j}}\lambda_{\mathfrak{h},j-1} + \sum_{p=j}^{\ell} \frac{\lambda_{\mathfrak{h},p}}{\sqrt{p(p+1)}} + \sqrt{\frac{\ell}{\ell-1}}\lambda_{\mathfrak{h},\ell-1} & \text{if } j>1 \text{ and } \ell>2 \\ \sum_{p=j}^{\ell} \frac{1}{\sqrt{p(p+1)}}\lambda_{\mathfrak{h},p} + \sqrt{\frac{\ell}{\ell-1}}\lambda_{\mathfrak{h},\ell-1} & \text{if } j=1 \text{ and } \ell>2 \\ \sqrt{\frac{\ell}{\ell-1}}\lambda_{\mathfrak{h},\ell-1} & \text{if } \ell=2 \end{cases} \tag{26}$$

For $(j\ell) \neq (pq)$:

$$\begin{aligned} [\lambda_{\mathfrak{k},\Re,j\ell},\, \lambda_{\mathfrak{k},\Re,pq}] =& \frac{i}{\sqrt{2}}\left(\delta_{\ell p}\lambda_{\mathfrak{k},\Im,jq} + \delta_{jp}\lambda_{\mathfrak{k},\Im,\ell q}\right. \\ &\left. + \delta_{jq}\lambda_{\mathfrak{k},\Im,\ell p} + \delta_{lq}\lambda_{\mathfrak{k},\Im,jp}\right) \\ [\lambda_{\mathfrak{k},\Re,j\ell},\, \lambda_{\mathfrak{k},\Im,pq}] =& \frac{i}{\sqrt{2}}\left(-\delta_{\ell p}\lambda_{\mathfrak{k},\Re,jq} - \delta_{jp}\lambda_{\mathfrak{k},\Re,\ell q}\right. \\ &\left. + \delta_{jq}\lambda_{\mathfrak{k},\Re,\ell p} + \delta_{lq}\lambda_{\mathfrak{k},\Re,jp}\right) \\ [\lambda_{\mathfrak{k},\Im,j\ell},\, \lambda_{\mathfrak{k},\Im,pq}] =& \frac{i}{\sqrt{2}}\left(-\delta_{\ell p}\lambda_{\mathfrak{k},\Im,jq} + \delta_{jp}\lambda_{\mathfrak{k},\Im,\ell q}\right. \\ &\left. - \delta_{jq}\lambda_{\mathfrak{k},\Im,\ell p} + \delta_{lq}\lambda_{\mathfrak{k},\Im,jp}\right) \end{aligned} \tag{27}$$

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