TSRT09 – Control Theory

Lecture 3: Disturbance models

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Summary of lecture 2

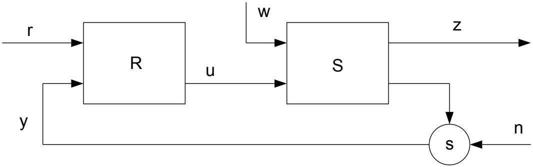
- Most of the theory for SISO systems generalizes to MIMO
- Significant differences exist in
  - minimal realization
  - controllability and observability canonical forms
  - computation of poles and zeros in transfer function matrices
- Poles:
  1. Eigenvalues of a state matrix (for a minimal realization)
  2. Roots of the *pole polynomial* = least common multiple of all denominators of all minors (i.e., determinant of square submatrices) of $G(s)$.
- Zeros:
  1. Roots of the *zero polynomial* = greatest common divisor for the numerators of the maximal minors of $G(s)$

Lecture 3

- Description of disturbances
- Standard form of system with disturbances in state space form
- Observer
- Kalman filter

In the book: Ch. 5

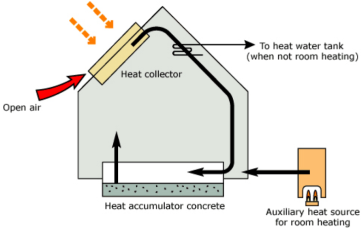
What is a disturbance?



Two main types of disturbances

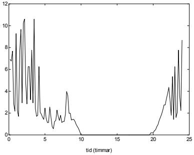
1. System disturbance: w
   - affect the system to be controlled
   - variations in the process
   - nonmodeled dynamics
   - extra non-controllable inputs
2. Measurement disturbance / noise: n
   - affects y but not z
   - measurement noise in the sensors
   - more general sensors disturbances

Example: solar heated house



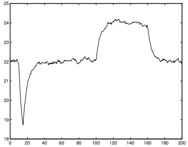
System disturbances:

- solar irradiation
- external temperature, wind
- are the windows open?
- how many people in the house?



Measurement disturbance / error

- sensor round-off
- gradient of temperature across different rooms



Models for disturbances

Adding disturbances to our models:

- in transfer functions:

$$Y(s) = G(s)\big(U(s) + W(s)\big) + N(s)$$

- in state space models

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + Nw(t) \\ y(t) &= Cx(t) + Du(t) + n(t) \end{aligned}$$

Models for the disturbances

- Time domain: covariance
- Frequency domain: spectrum

Signal size measure

- Our signal size measure:

$$\|z\|_2^2 = \int_{-\infty}^{\infty} |z(t)|^2 \; dt \qquad |z|^2 = z^T z \qquad \text{“energy”}$$

- If the integral does not converge, one can use the measure

$$\|z\|_e^2 = \lim_{N \to \infty} \frac{1}{N} \int_0^N |z(t)|^2 \; dt \qquad \text{“power”}$$

This is a rough description (a scalar)

More informative measure for vectors of signals

- How do the components of z relate to each other? Matrix measure

$$R_z = \int_{-\infty}^{+\infty} z(t)z^T(t)\ dt \qquad \text{“energy”}$$

$$R_z = \lim_{N\to\infty} \frac{1}{N}\int_0^N z(t)z^T(t)\ dt \qquad \text{“power”}$$

- More general: “covariance (at τ)”

$$R_z(\tau) = \int_{\infty}^{+\infty} z(t)z^T(t-\tau)dt$$

- $\implies R_z =$ “covariance (at 0)”

$$R_z = R_z(0)$$

Underlying idea: stationary stochastic processes

- at each t: $z(t)$ = vector of random var. $\Longrightarrow$ stochastic process
- stationary stochastic process: description does not depend on t
- to describe a stochastic process:
  1. mean
     $$m_z = E[z(t)]$$
     "mathematical expectation"
  2. variance
     $$\mathrm{Var}[z(t)] = E[(z(t) - m_z)^2]$$
     "deviation from the mean"
  3. covariance
     $$R_z(\tau) = E[(z(t) - m_z)(z(t-\tau) - m_z)^T]$$
     matrix, "signal compared with itself, but shifted"
- for us: zero mean $m_z = 0$

Autocovariance vs. cross-covariance

Example

$$z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \quad \Longrightarrow \quad R_z(\tau) = \begin{bmatrix} R_{z_1,z_1}(\tau) & R_{z_1,z_2}(\tau) \\ R_{z_2,z_1}(\tau) & R_{z_2,z_2}(\tau) \end{bmatrix}$$

- diagonal

$$R_{z_i,z_i}(\tau) = \int_{-\infty}^{+\infty} z_i(t) z_i(t-\tau) \; dt$$

"autocovariance"

- off-diagonal

$$R_{z_i,z_j}(\tau) = \int_{-\infty}^{+\infty} z_i(t) z_j(t-\tau) \; dt$$

cross-covariance

$R_{z_i,z_j} = 0 \iff$ z_i and z_j are uncorrelated

Spectrum = "size distribution" at different frequencies

Spectrum of $z(t)$: $\Phi_z(\omega) =$ "Energy content of z at frequency ω"

- $\Phi_z(\omega) =$ matrix function
- if $Z(i\omega) =$ Fourier transform of $z(t)$

$$\Phi_z(\omega) = Z(i\omega)Z^*(i\omega)$$

- using Parceval

$$R_z = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Phi_z(\omega)d\omega$$

Spectrum

- Alternatively: $\Phi_z(\omega)$ Fourier transform of $R_z(\tau)$

$$\Phi_z(\omega) = \int_{-\infty}^{\infty} R_z(\tau) e^{-i\omega\tau} \, d\tau$$

- $R_z(\tau)$ and $\Phi_z(\omega)$ form a Fourier transform pair

$$\Phi_z(\omega) = \mathcal{F}[R_z(\tau)], \qquad R_z(\tau) = \mathcal{F}^{-1}[\Phi_z(\omega)]$$

- when computed in $\tau = 0$:

$$R_z = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi_z(\omega) d\omega$$

Cross spectrum

Example

$$z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \quad \Longrightarrow \quad \Phi_z(\omega) = \begin{bmatrix} \Phi_{z_1,z_1}(\omega) & \Phi_{z_1,z_2}(\omega) \\ \Phi_{z_2,z_1}(\omega) & \Phi_{z_2,z_2}(\omega) \end{bmatrix}$$

- Diagonal element $\Phi_{z_i,z_i}(\omega)$ measures the energy content of z_i at frequency ω
- Off-diagonal element $\Phi_{z_i,z_j}(\omega)$ measures the coupling between z_i and z_j at frequency ω
  - $\Phi_{z_i,z_j}(\omega) =$ cross spectrum between z_i and z_j

$$\Phi_{z_i,z_j}(\omega) = \int_{-\infty}^{\infty} R_{z_i,z_j}(\tau) e^{-i\omega\tau}\, d\tau$$

  - $\Phi_{z_i,z_j}(\omega) \equiv 0 \implies z_i$ and z_j are uncorrelated ($\iff R_{z_i,z_j} = 0$)

White noise = unpredictable signal

White noise = signal with constant spectrum

$$\Phi_z(\omega) = R = \text{ const} \quad \forall\omega$$

- Covariance:

$$R_z(\tau) = \delta(\tau)R = \begin{cases} R & \text{if } \tau = 0 \\ 0 & \text{if } \tau \neq 0 \end{cases}$$

- Meaning: old values give no information on the future values
- $\Longrightarrow$ white noise is unpredictable

Spectra and transfer functions

If u has spectrum $\Phi_u(\omega)$, what is $\Phi_y(\omega) =$ spectrum of y?

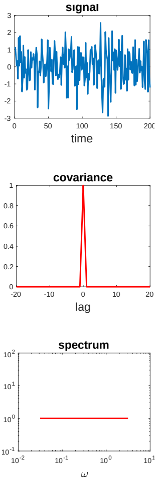
$$Y(i\omega) = G(i\omega)U(i\omega), \quad U, Y \text{ Fourier transforms}$$

$$\Phi_y(\omega) = Y(i\omega)Y(i\omega)^* = G(i\omega)U(i\omega)U(i\omega)^* G(i\omega)^*$$

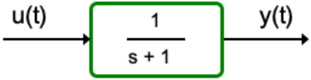
or

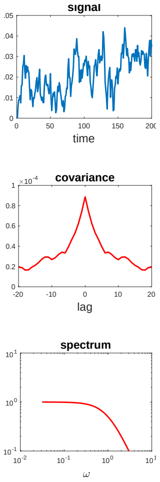
$$\boxed{\Phi_y(\omega) = G(i\omega)\Phi_u(\omega)G(i\omega)^*}$$

Example of spectrum

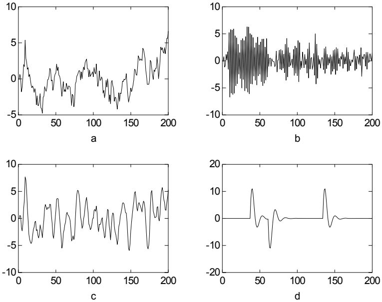


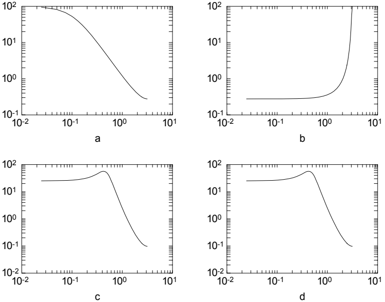
$$Y(s) = \frac{1}{s+1} U(s)$$





A few "colored noises" and their spectra





$$u \in \mathcal{N}(0,\ \lambda) \qquad u = \begin{cases} 0 & \text{with prob. } 0.8 \\ \in \mathcal{N}(0,\ \lambda) & \text{with prob. } 0.2 \end{cases}$$

Disturbance from white noise: spectral factorization

"Reverse problem": given $y(t)$ of spectrum $\Phi_y(\omega)$, choose $G(s)$ and a white noise $u(t)$ of spectrum $\Phi_u(\omega) = R$ so that

$$Y(s) = G(s)U(s) \qquad \text{and} \qquad \Phi_y(\omega) = G(i\omega)RG(i\omega)^*$$

$\Longrightarrow$ spectral factorization

- Scalar case. If u and y are scalar and Φ_y is a rational function: easy to solve with G stable.

- Matrix case. If u and/or y are matrices: more difficult

State space model with disturbances

- state space model with disturbances

$$\dot{x}(t) = Ax(t) + Bu(t) + Nw(t)$$
$$y(t) = Cx(t) + Du(t) + n(t)$$

- $w(t)$ and $n(t)$ are in general not white $\implies$ "colored noise"

- Task: model should contain all information useful to predict future values $\implies$ also "colored noise" contributes

- "Whitening" the disturbances: w and n are outputs of the linear systems $w = G_w v_1$, $n = G_n v_2$, where v_1, v_2 are white

State space model with disturbances

- Expressing G_w, G_n in state space form

$$\dot{\bar{x}}(t) = \bar{A}\bar{x}(t) + \bar{B}u(t) + \bar{N}v_1(t)$$
$$y(t) = \bar{C}\bar{x}(t) + Du(t) + v_2(t)$$

$\Longrightarrow \bar{x}$ is extended to include the states in G_w, G_n

$\Longrightarrow$ "new" disturbance v_1 and measurement noise v_2 are white noises $\Longrightarrow$ unpredictable

"Size" of x induced by white noise

$$\dot{x} = Ax + Nv_1, \qquad \Phi_{v_1}(\omega) = R$$

- Transfer function from v_1 to x is $(sI - A)^{-1}N$
- Spectrum is then

$$\Phi_x(\omega) = (i\omega I - A)^{-1} NRN^T (-i\omega I - A)^{-T}$$

- "Size" of x (i.e., covariance matrix of x)

$$\Pi_x = R_x(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi_x(\omega) d\omega$$

Theorem: Π_x is the solution to the Lyapunov equation

$$A\Pi_x + \Pi_x A^T + NRN^T = 0$$

State estimation: observer

$$\dot{x}(t) = Ax(t) + Bu(t) + Nv_1(t)$$
$$y(t) = Cx(t) + Du(t) + v_2(t)$$

- **Task**: estimate x from the measurement y
- $\implies$ state observer $\hat{x}$
- Dynamics of $\hat{x}$:

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + K\big(y(t) - C\hat{x}(t) - Du(t)\big)$$

State estimation: observer

- Estimation error: $\tilde{x}(t) = x(t) - \hat{x}(t)$

- Dynamics of $\tilde{x}$:

$$\dot{\tilde{x}}(t) = (A - KC)\tilde{x}(t) + Nv_1(t) - Kv_2(t)$$

- Choice of K decides how fast the estimation error converges and also how much the estimation is influenced by the measurement error $\implies$ Trade off

Kalman filter: the optimal observer

$$K = PC^T R_2^{-1}$$

where P is determined from

$$AP + PA^T - PC^T R_2^{-1} CP + NR_1 N^T = 0 \qquad \text{Riccati equation}$$

- Optimality 1: The linear filter that has the least mean square error (for all noise for which the covariance is defined)
- Optimality 2: (for Gaussian noise) The best (linear or nonlinear) filter for many criteria. $\hat{x}$ is the conditional expectation of x, given the observations.

Disturbance: something "kicking the system"

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