Institutionen för Reglerteknik
Tekniska Högskolan i Lund

Linear Systems I

Exam March 6-17, 1995

Solutions to all problems should be well motivated. There is a total of 59 points, including 14 from the hand in problems. At least 30 should be reached for passed exam.

The examination time is 48 hours. Computers may be used and books may be consulted (except for the book where the appendix on four wheel car steering originates). You are encouraged to ask me if anything is questionable or difficult to understand, but you may not use help from each other.

I am grateful for your feedback on the course and would also be happy to have your errata collection for the book.

Good Luck!
Anders

1. The transfer matrix can be rewritten as

$$\frac{1}{s+1}\begin{bmatrix} 0 & 0 & 4 \\ 3 & 1 & 0 \end{bmatrix} + \frac{1}{s+2}\begin{bmatrix} 1 \\ 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & -6 \end{bmatrix}$$
$$+\frac{1}{s+3}\begin{bmatrix} 1 & 1 & 2 \\ -3 & -3 & 1 \end{bmatrix} + \frac{1}{s+5}\begin{bmatrix} -1 \\ 2 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

The following minimal realization can therefore be obtained as in Rugh's exercise 10.11, which was solved in session 5.

$$\dot{x} = \begin{bmatrix} -1 & & & & & \\ & -1 & & & 0 & \\ & & -2 & & & \\ & & & -3 & & \\ & 0 & & & -3 & \\ & & & & & -5 \end{bmatrix} x + \begin{bmatrix} 0 & 0 & 4 \\ 3 & 1 & 0 \\ 0 & 0 & -6 \\ 1 & 1 & 2 \\ -3 & -3 & 1 \\ 1 & 0 & 0 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 & 2 \end{bmatrix} x$$

2. All eigenvalues of are inside the unit circle, so there exists a transformation matrix T with $\|T^{-1}AT\| = \rho < 1$. Hence the convergence of the series follows from the bound

$$\left\| \sum_{k=0}^{N} A^k BB^T (A^k)^T \right\| \leq \sum_{k=0}^{N} \|T(T^{-1}AT)^k T^{-1}BB^T T^{-T}(T^T A^T T^{-T})^k T^T\|$$
$$\leq \sum_{k=0}^{N} \|T\|^2 \|T^{-1}BB^T T^{-1}\| \rho^N \leq \frac{\|T\|^2 \|T^{-1}BB^T T^{-1}\|}{1-\rho^2}$$

a. The desired equality follows immediately as

$$APA^T = A\left(\sum_{k=0}^{\infty} A^k BB^T (A^k)^T\right) A^T = \sum_{k=0}^{\infty} A^{k+1}BB^T(A^{k+1})^T$$
$$= \sum_{k=1}^{\infty} A^k BB^T (A^k)^T = P - BB^T$$

b. Define the linear operator L_n by

$$L_n u = \sum_{k=0}^{n-1} A^{n-k-1} Bu(k)$$

Then $x(n) = L_n(u)$ and we need to show that every element of $\mathcal{R}(P)$ also is in $\mathcal{R}(L_N)$ for some N. Note that

$$L_n L_n^* = \sum_{k=0}^{n} A^k BB^T (A^k)^T$$
$$L_0 L_0^* \leq L_1 L_1^* \leq L_2 L_2^* \ldots \leq P$$

Hence $\dim \mathcal{N}(L_n L_n^*)$ is a decreasing sequence of positive integers and for sufficiently large N, we have $\mathcal{N}(L_N L_N^*) = \mathcal{N}(P)$. For such N

$$\mathcal{R}(P) = \mathcal{N}(P)^\perp = \mathcal{N}(L_N L_N^*)^\perp = \mathcal{N}(L_N)^\perp = \mathcal{R}(L_N)$$

and the proof is complete.

3. With $z = [\dot{x}\ x]^T$, the equation becomes

$$\dot{z} = A(t)z(t)$$

where

$$A(t) = \begin{bmatrix} -1 - \cos\omega t & -1 \\ 1 & 0 \end{bmatrix}$$

For any $\omega \in R$, Theorem 7.2 in Rugh can be applied with $Q = I$. This shows uniform stability, since

$$A(t)^T Q + QA(t) = A(t)^T + A(t) = \begin{bmatrix} -2 - 2\cos\omega t & 0 \\ 0 & 0 \end{bmatrix} \leq 0$$

4.

a. The map from the initial state x_0 and the input u to the output y can be written $y = L_1 x_0 + L_2 u$, where

$$(L_1 x_0)(t) = C(t)\Phi(t, 0)x_0$$
$$(L_2 u)(t) = C(t) \int_0^t \Phi(t, s)B(s)u(s)ds$$

The formula for the least squares estimate of x_0, based on y and u, is

$$\hat{x}_0 = (L_1^* L_1)^{-1} L_1^*(y - L_2 u)$$

The adjoint L_1^* can be determined from the identity

$$< z, L_1 x_0 > = \int_0^T z(t)^T C(t)\Phi(t, 0)x_0 dt =< L_1^* x, x_0 >$$

where

$$L_1^* z = \int_0^T \Phi(t, 0)^T C(t)^T z(t)dt$$
$$L_1^* L_1 = \int_0^T \Phi(t, 0)^T C(t)^T C(t)\Phi(t, 0)dt = M(0, T)$$

Hence

$$\hat{x}_0 = M(0, T)^{-1} \int_0^T \Phi(t, 0)^T C(t)^T z(t)dt$$
$$z(t) = y(t) - C(t) \int_0^t \Phi(t, s)B(s)u(s)ds$$

solves the least squares problem.

b. We have

$$\begin{aligned}\hat{x}_0 &= M(0,t)^{-1}\int_0^T \Phi(t,0)^T C(t)^T [y - L_2 u](t)dt \\ &= M(0,T)^{-1}\int_0^T \Phi(t,0)^T C(t)^T [e(t) + C(t)\Phi(t,0)x_0]dt \\ &= M(0,T)^{-1}\int_0^T \Phi(t,0)^T C(t)^T e(t)dt + x_0\end{aligned}$$

$$\begin{aligned}|\hat{x}_0 - x_0|^2 &= \int_0^T\int_0^T e(t)^T C(t)\Phi(t,0)M(0,T)^{-2}\Phi(s,0)^T C(s)^T e(s)dsdt \\ &= \int_0^T\int_0^T e(t)^T W(t,s)e(s)dsdt\end{aligned}$$

with

$$W(t,s) = C(t)\Phi(t,0)M(0,T)^{-2}\Phi(s,0)^T C(s)^T.$$

c. The formulas in **a** and **b** give

$$\begin{aligned}\hat{x}_0 &= \begin{bmatrix}(1-e^{-2T})/2 & (1-e^{-3T})/3 \\ (1-e^{-3T})/3 & (1-e^{-4T})/4\end{bmatrix}^{-1}\int_0^T \begin{bmatrix}e^{-t} \\ e^{-2t}\end{bmatrix} z(t)dt \\ z(t) &= y(t) - \int_0^t (e^{s-t} + e^{2(s-t)})u(s)ds \\ W(t,s) &= \begin{bmatrix}e^{-t} & e^{-2t}\end{bmatrix}\begin{bmatrix}(1-e^{-2T})/2 & (1-e^{-3T})/3 \\ (1-e^{-3T})/3 & (1-e^{-4T})/4\end{bmatrix}^{-2}\begin{bmatrix}e^{-s} \\ e^{-2s}\end{bmatrix}\end{aligned}$$

d. The relationship $(L_1^* L_1)^{-1}L_1^* e = (\hat{x}_0 - x_0)$ from **b**, together with the least squares formula

$$\min_{Ae=b} |e|^2 = b^*(AA^*)^{-1}b$$

gives

$$|e|^2 = (\hat{x}_0 - x_0)^* L_1^* L_1(\hat{x}_0 - x_0) = (\hat{x}_0 - x_0)^* M(0,T)(\hat{x}_0 - x_0)$$

With $|\hat{x}_0 - x_0|^2$ fixed to ϵ, the norm of e is therefore minimal if $\hat{x}_0 - x_0$ is chosen as an eigenvector of $M(0,T)$ corresponding to the smallest eigenvalue. If the system is close to unobservable, in the sense that $M(0,T)$ is close to singular, even a small disturbance e chosen this way, may cause a large observation error $|\hat{x}_0 - x_0|$.

e. The formulas in **a** and **b** give

$$\begin{aligned}\hat{x}_0 &= \frac{1}{2\pi}\int_0^{4\pi}\begin{bmatrix}\sin t \\ \cos t\end{bmatrix} z(t)dt \\ z(t) &= y(t) - \begin{bmatrix}\sin t & \cos t\end{bmatrix}\int_0^t u(s)ds \\ W(t,s) &= \frac{1}{4\pi}\begin{bmatrix}\sin t & \cos t\end{bmatrix}\begin{bmatrix}\sin s \\ \cos s\end{bmatrix} \\ &= \frac{1}{4\pi}\cos(t-s)\end{aligned}$$

5. This is essentially a problem of noninteracting control, the main difference being that the matrix N is fixed to identity. Considering a_f and r as outputs, the matrix Δ in Theorem 14.12 is

$$\Delta = \begin{bmatrix} L_A^0[C_{a_f}]B \\ L_A^0[C_r]B \end{bmatrix} = \begin{bmatrix} d_1 & c_1 b_{12} + c_2 b_{22} \\ 0 & b_{22} \end{bmatrix}$$

By Rugh, Lemma 14.11, $L_A^0[C_{a_f}] = L_{A+BK}^0[C_{a_f}]$ regardless of K, so in order for a_f not to be controllable from u_r, it is necessary to assume that

$$c_1 b_{12} + c_2 b_{22} = 0$$

(In fact, this identity always holds for a vehicle model consisting of two point masses, one at the front axis and one at the rear axis.) Then Δ is diagonal, so N is not needed and the formula (45) for noninteracting control

$$\begin{aligned} K &= -\Delta^{-1}[\Omega A + \dot{\Omega}] \\ &= -\begin{bmatrix} d_1 & 0 \\ 0 & b_{22} \end{bmatrix}^{-1} \begin{bmatrix} c_1 & c_2 & d_1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & b_{11} \\ a_{21} & a_{22} & b_{21} \\ 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

solves the problem.