SYSTEM IDENTIFICATION IN A NOISE-FREE ENVIRONMENT

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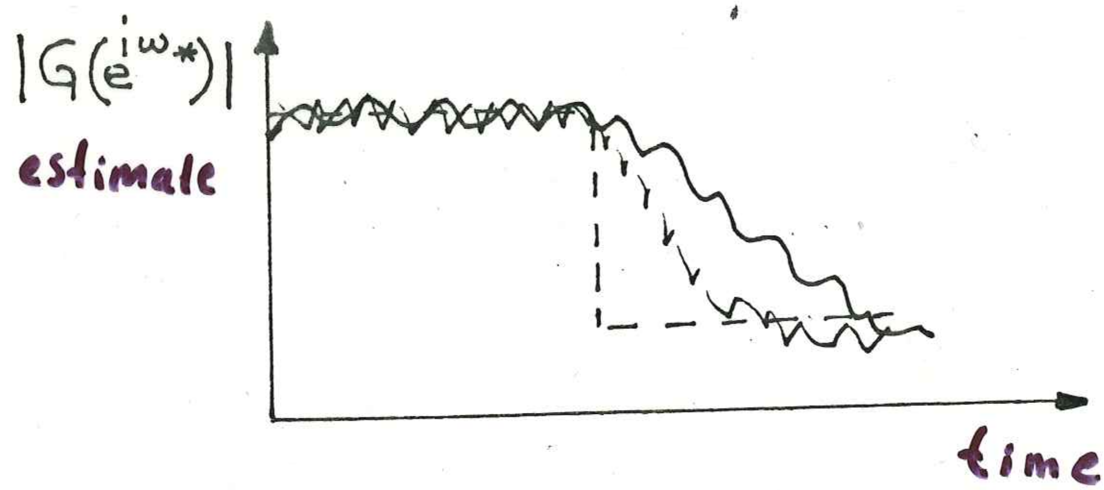
THE PROBLEM

THE WORLD CHANGES -

ADAPT TO IT QUICKLY



- - - - True system

______ Estimate

THINGS TAKE TIME!

- WHY DO THINGS TAKE TIME?
  - measurements are not reliable
  - noise effects must be averaged out over time
  - trade-off between noise sensitivity and tracking ability
- WHAT IF (VIRTUALLTY) NO NOISE?

THE QUESTIONS

- Noise free system: $y(t) = G_0(q)\, u(t)$
- Apply standard (recursive) prediction error method
- Design variables
  - Forgetting factor
  - Data prefilter

- What are the properties of the estimates?
- How do they depend on forgetting factor and prefilter?
- What are the trade-offs for quick adaptation?

THE MACHINERY I

- SYSTEM: $y(t) = G_0(q)\, u(t)$

$$G_0(q)u(t) = \left(\sum_{k=1}^{\infty} g_k q^{-k}\right) u(t) = \sum_{k=1}^{\infty} g_k u(t-k)$$

- FREQUENCY FUNCTION: $G_0(e^{i\omega})$

- MODEL:

$$y(t) = G(q,\theta)u(t) + H(q,\theta)e(t)$$

THE MACHINERY II

- FORM PREDICTION ERRORS

$$\varepsilon(t,\theta) = H^{-1}(q,\theta)[y(t) - G(q,\theta)u(t)]$$

- FILTER DATA:

$$\varepsilon_F(t,\theta) = L(q)\,\varepsilon(t,\theta)$$

- CHOOSE FORGETTING FACTOR λ

- MINIMIZE

$$\sum_{t=1}^{N} \lambda^{N-t} \varepsilon_F^2(t,\theta)$$

GIVES $\hat{\theta}_N$

OUTLINE

- THE PROBLEM
- THE QUESTIONS
- THE MACHINERY
- THE ANSWERS
    - Asymptotic properties for $\lambda \to 1$
    - How to explain variability
    - Prefilter and variability
    - Transient behaviour
    - Quick adaptation
- THE CONCLUSIONS

ASYMPTOTIC PROPERTIES AS $\lambda \to 1$

$$\hat{\Theta}_N \to \Theta^*$$

$$\Theta^* = \arg\min_{\theta} \int_{-\pi}^{\pi} \left| G_0(e^{i\omega}) - G(e^{i\omega}, \theta) \right|^2 \frac{\Phi_u(\omega) \cdot |L(e^{i\omega})|^2}{|H(e^{i\omega}, \theta)|^2} d\omega$$

$\Phi_u(\omega)$ input spectrum

$$R_u(\tau) = \lim_{N \to \infty} \frac{1}{N} \sum_{t=1}^{N} u(t+\tau) u(t)$$

$$\Phi_u(\omega) = \sum_{\tau=-\infty}^{\infty} R_u(\tau) e^{-i\tau\omega)}$$

ASYMPTOTIC PROPERTIES AS $\lambda \to 1$

If $\{u(t)\}$ stochastic process

$$(\hat{\Theta}_N - \Theta^*) \in As\ N(0, P)$$

$$* \; P = \frac{(1-\lambda)}{2} \cdot Q \qquad (\text{If } \lambda = 1,\ \frac{1}{N} Q)$$

$$Q = R^{-1} S R^{-1}$$

$$R = E\psi(t)\psi^T(t); \quad \psi(t) = -\frac{d}{d\theta}\varepsilon_F(t,\theta)\mid_{\theta=\theta^*}$$

$$S = \lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N}\sum_{s=1}^{N} E\psi(t)\psi^T(s)\varepsilon(t)\varepsilon(s)$$

$$\varepsilon(t) = \varepsilon_F(t, \theta^*)$$

ASYMPTOTIC PROPERTIES cont.

CONCEPTUALLY:

$$\text{Variance } P \sim (1 - \lambda) \cdot \frac{\text{Energy in } \varepsilon}{\text{Energy in } \Psi}$$

NOTE:

- Decays as if noise were present
- Model error plays the role of noise
- Variance - Variability

$|G(e^{+i\omega_*}, \hat{\theta}_t)|$ as a function of t

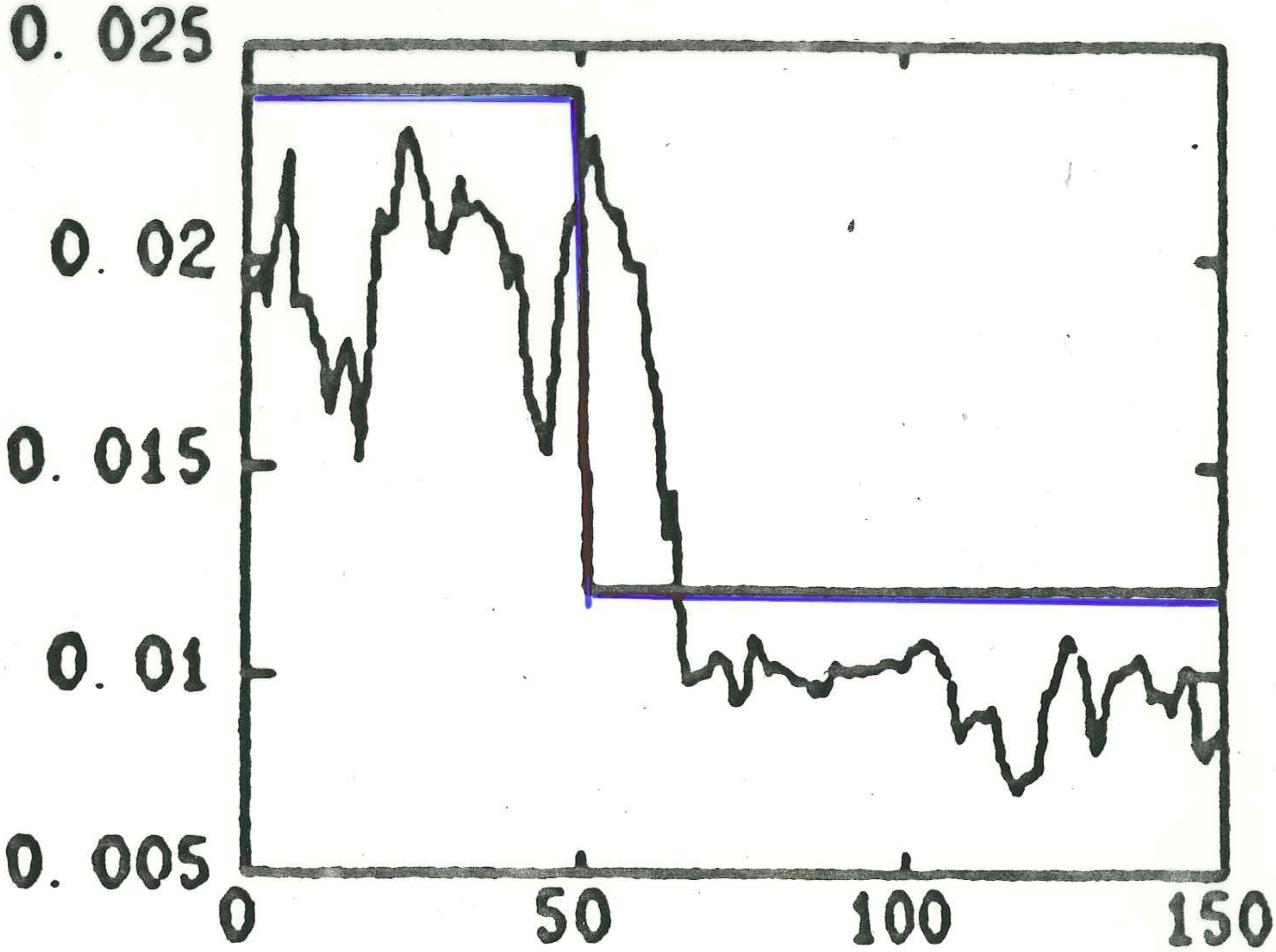


Figure 13: As Figure 11, but $\lambda = 0.8$.

HOW TO EXPLAIN VARIABILITY?

- True system $G_0(e^{i\omega})$

Consider model

$y(t) + a\, y(t-1) = bu(t-1)$

a & b can be fitted using

$y(t_1) + a\, y(t_1-1) = b\, u(t_1-1)$

$y(t_1+1) + a\, y(t_1) = b\, u(t_1)$

What is the resulting model?

$$\frac{be^{-i\omega_* t}}{1 + ae^{-i\omega_* t}} = G_0(e^{i\omega_*}) \quad !$$

for ω_* being the unique frequency that fits

$y(t_1-1), y(t_1), y(t_1+1)$

- Exact fit at "instantaneous frequency"

- Normally $\hat{G}_t(e^{i\omega_0})$ will vary.

When no variability?

- Instantaneous frequency time-independent

or

- True system first order

- CONCEPTUALLY:

The model varies since it tries to adjust to the true system in the frequency range reflected by the recent inputs. This range fluctuates, though, even for a "stationary" input.

VARIABILITY OF TRANSFER FUNCTION

Translated to the transfer function estimate, for high order models

limit model   model order   true system

$$E\left|\hat{G}_N(e^{i\omega}) - G^*(e^{i\omega})\right|^2 \sim \frac{n}{2}(1-\lambda)\left|G_0(e^{i\omega}) - G^*(e^{i\omega})\right|^2$$

estimate   forgetting factor

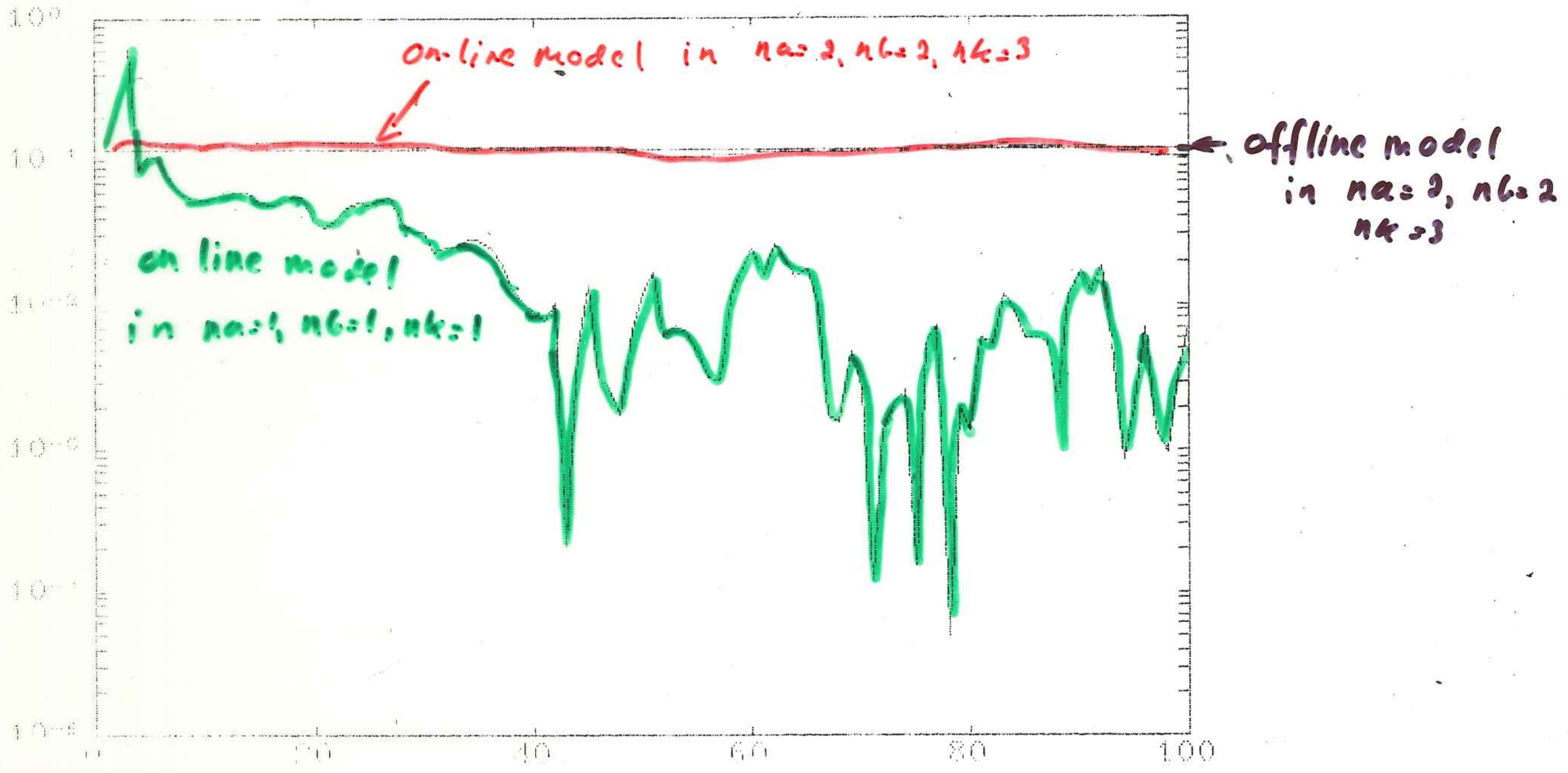
(Formally correct for FIR model with white noise input)

- IMPLICATIONS!

"HAIRDRYER DATA"

$\lambda = 0.95$

$|G(e^{i \cdot 1})|$ 12.5 rad/s



PREFILTER AND VARIABILITY

* Prefilter will affect limit model.

Recall

* $$Q \sim \frac{\text{energy in } \varepsilon_F}{\text{energy in } \Psi_F}$$

* Can Q be reduced by prefilter?

Well ...

- Let L(q) be a narrow band ideal BP-filter with BW W.

Then

- energy in $\Psi_F \sim C_1 \cdot W$

$$\varepsilon_F = L(q)H^{-1}(q,\theta^*)[G_0(e^{i\omega}) - G_*(e^{i\omega})]u(t)$$

energy in $\varepsilon_F =$

$$\sim \int_{\omega_0 - W}^{\omega_0 + W} \left| G_*(e^{i\omega}) - G_0(e^{i\omega}) \right|^2 d\omega \sim$$

$$\sim \int_{\omega_0 - W}^{\omega_0 + W} \left| G_*(e^{i\omega_0}) - G_0(e^{i\omega_0}) + G' \cdot (\omega - \omega_0) \right|^2 d\omega \sim$$

$$\sim D_2 \cdot W^3$$

implies

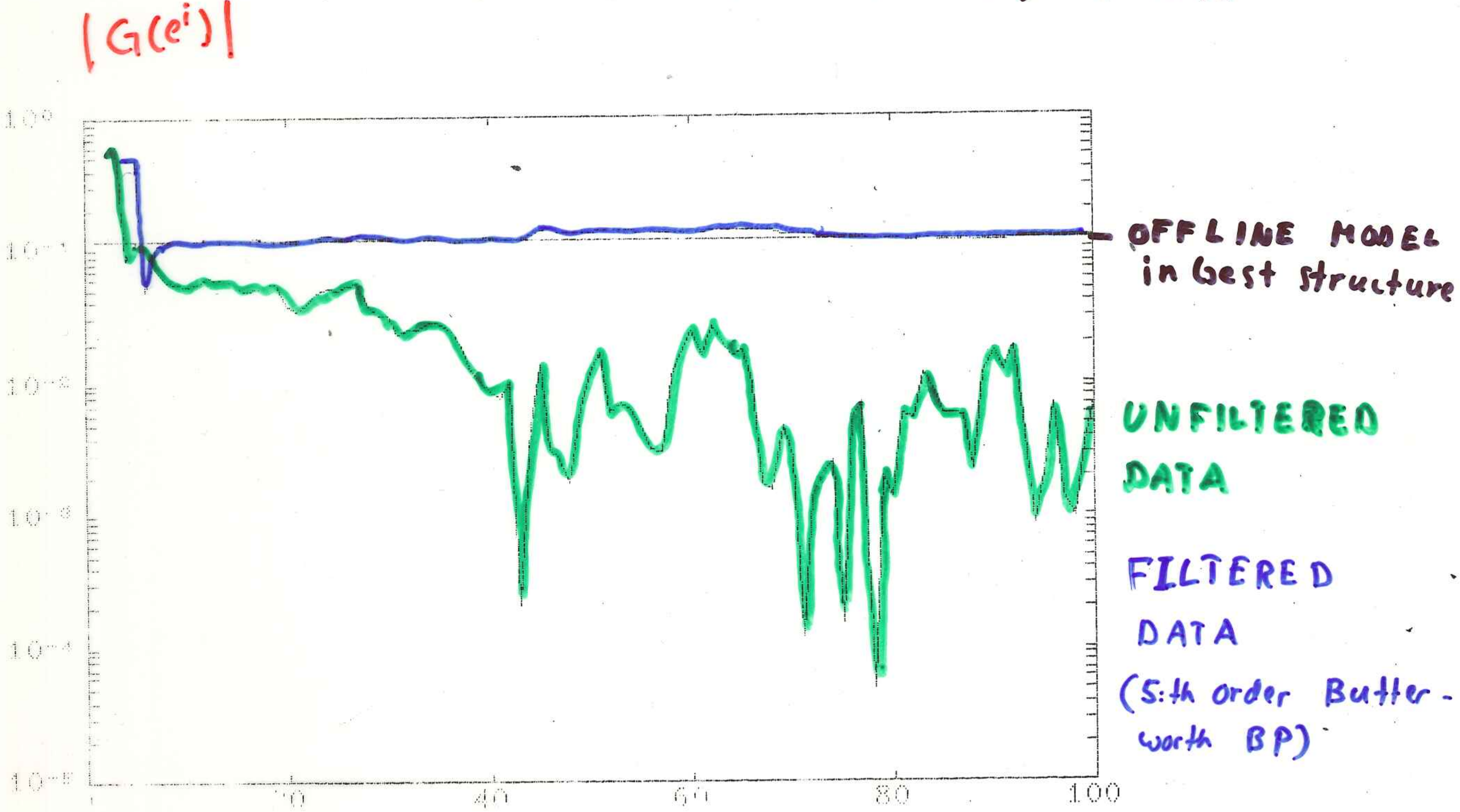
$$\boxed{Q \sim C \cdot W^2}$$

Variability will be reduced by prefiltering!

HAIRDRYER DATA

ON-LINE MODELS IN

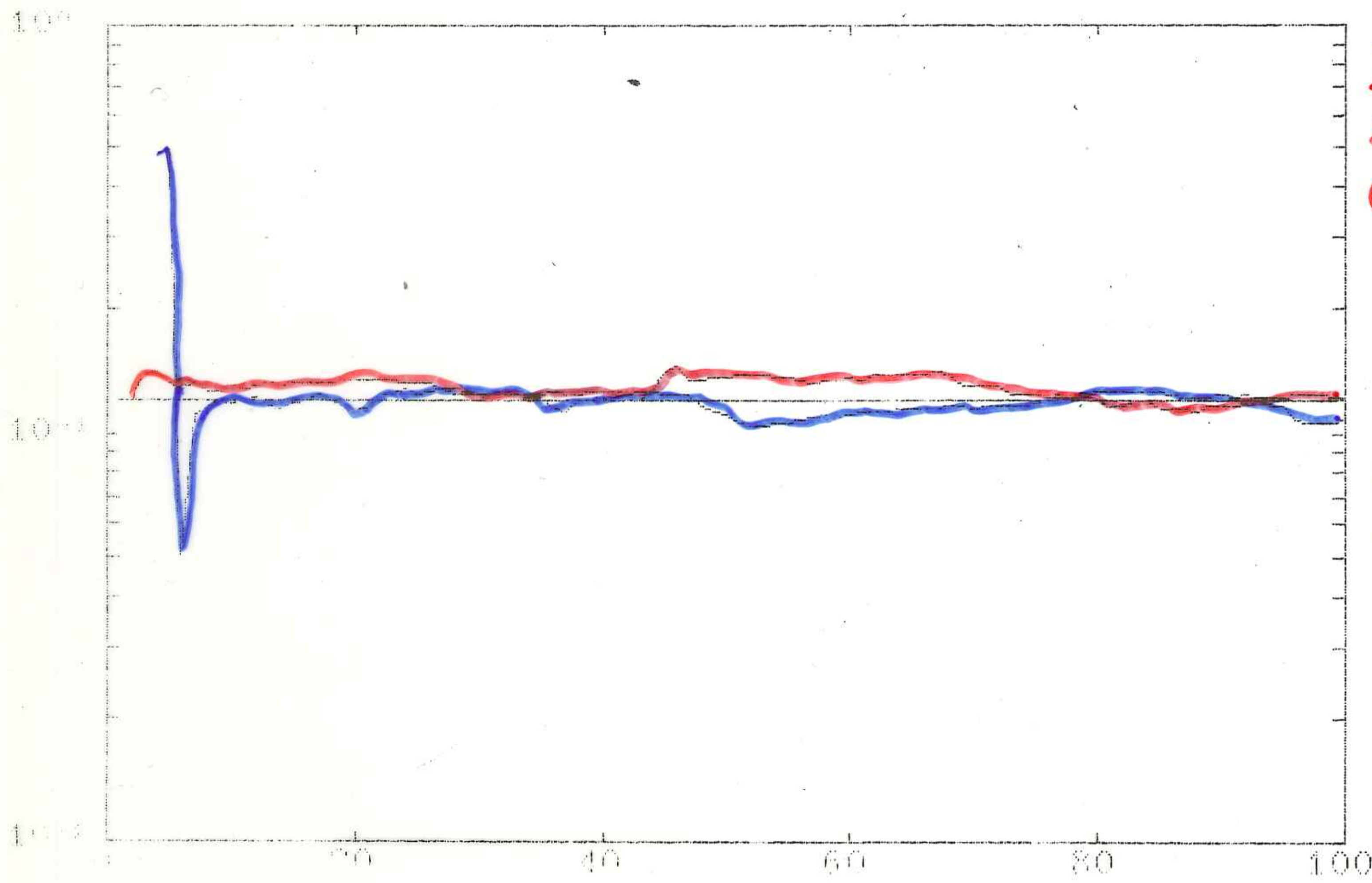
$na=1,\ nb=1,\ nk=1,\ \lambda=0.95$



HAIRDRYER DATA

$|G(e^{i})|$

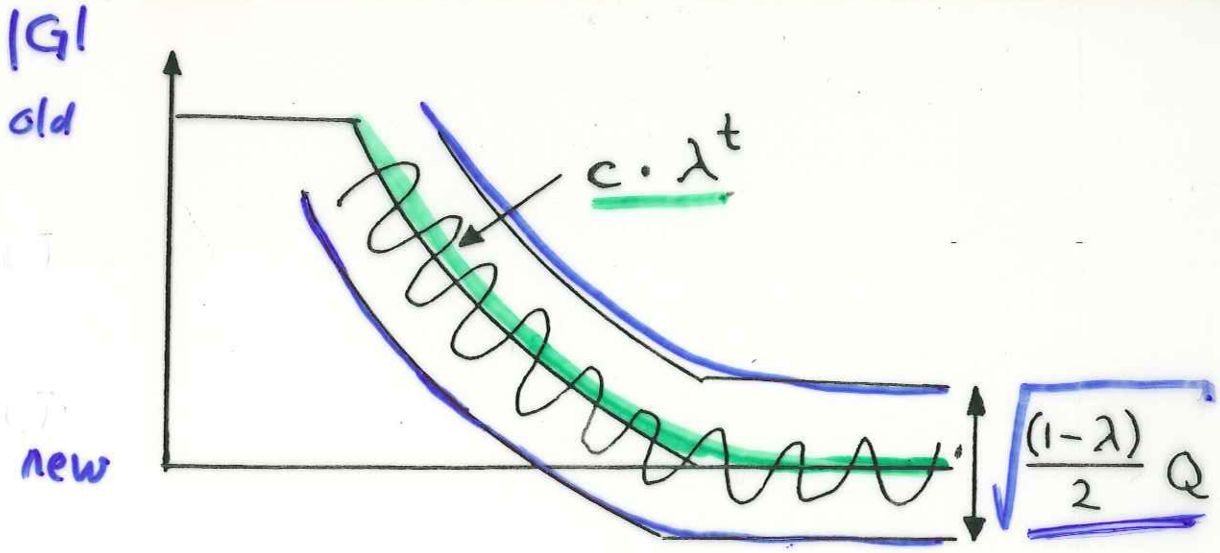
ON-LINE MODELS $\lambda = 0.95$



IN BEST STRUCTURE - UNFILTERED DATA

IN SIMPLEST STRUCTURE - FILTERED DATA

TRANSIENT BEHAVIOUR



- $Q \sim C \cdot W^2$
- Quick adaptation?
- What about prefilter?

FILTER BEHAVIOUR

n:th order BP Butterworth filter with bandwidth W has a transient

$$C \cdot \mu^t$$

$$\mu = e^{-W \cdot \sin\frac{\pi}{2n}} \approx \left(1 - W \cdot \frac{\pi}{2n}\right)$$

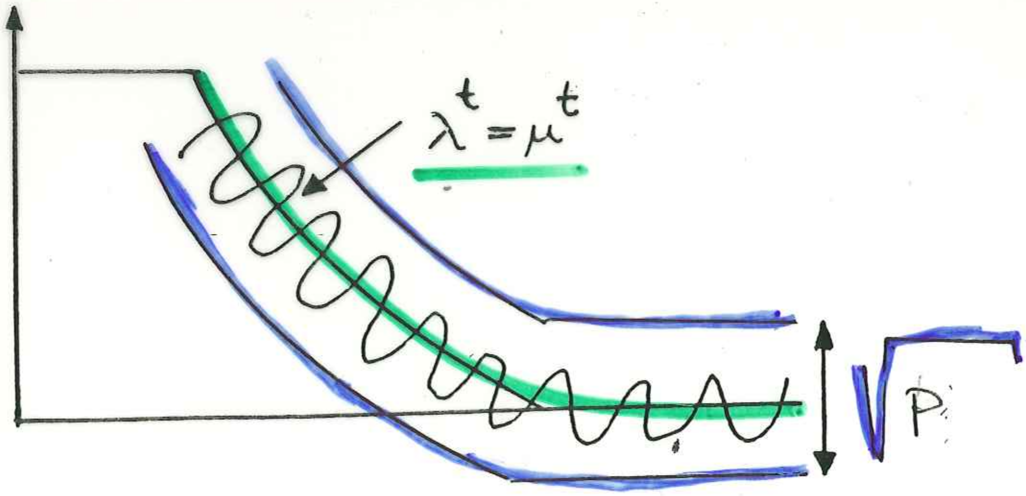
Narrow band/high order will delay information.

* LINK λ and filter so that

$$\lambda \approx \mu$$

$$\Rightarrow (1 - \lambda) \approx W \cdot \frac{\pi}{n}$$

QUICK ADAPTATION



$$P \sim (1-\lambda)Q = W \cdot \frac{\pi}{n} \cdot C \cdot W^2$$

$$\mu = (1 - \frac{W \cdot \pi}{n})$$

$$\min_t \left| \left(1 - \frac{W \cdot \pi}{n}\right)^t + \sqrt{\frac{W^3}{n}} \right| < \delta$$

CHOICE OF FORGETTING FACTOR

- Still a trade-off between estimate variability and tracking alertness

- Prefilter opens up important new dimensions

CONCLUSIONS

- ★ NO NOISE - STILL PROBLEM
- ★ MODEL ERROR TAKES UP THE PART OF THE NOISE
- ★ "SIGNAL-TO-NOISE RATIO" CAN BE IMPROVED BY PRE-FILTERING
- ★ MODEL VARIABILITY INDICATION OF MODEL ERROR
- ★ QUICK ADAPTATION: A JOINT VENTURE FOR FORGETTING FACTOR AND PREFILTER (AND MODEL COMPLEXITY!)