The Marginalized Auxiliary Particle Filter

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Abstract—In this paper we are concerned with nonlinear systems subject to a conditionally linear, Gaussian sub-structure. This structure is often exploited in high-dimensional state estimation problems using the marginalized (aka Rao-Blackwellized) particle filter. The main contribution in the present work is to show how an efficient filter can be derived by exploiting this structure within the auxiliary particle filter. Based on a multi-sensor aircraft tracking example, the superior performance of the proposed filter over conventional particle filtering approaches is demonstrated.

I. INTRODUCTION

Consider the following rather general discrete-time state-space model,

$$\mathbf{x}_k = \mathbf{f}_{k-1}(\mathbf{x}_{k-1}, \mathbf{w}_{k-1}), \tag{1a}$$

$$\mathbf{z}_k = \mathbf{h}_k(\mathbf{x}_k, \mathbf{e}_k), \tag{1b}$$

where k denotes the discrete-time index, $\mathbf{x}_k \in \mathbb{R}^{n_x}$ denotes the state vector, $\mathbf{z}_k \in \mathbb{R}^{n_y}$ denotes the measurement vector, $\mathbf{f}_{k-1}$ and $\mathbf{h}_k$ denote possibly time-varying functions. Finally, the process and measurement noise $\mathbf{w}_{k-1}$ and $\mathbf{e}_k$ are assumed to be mutually independent white noise sequences with known probability density functions (pdf's) $p_{\mathbf{w}}(\mathbf{w})$ and $p_{\mathbf{e}}(\mathbf{e})$. Here, it is worth noting that the extension of (1) to the case of multi-rate sensors is straightforward.

The aim in nonlinear filtering is to sequentially compute estimates of the state $\mathbf{x}_k$ using the sequence of all available measurements $\mathbf{Z}_k = \{\mathbf{z}_i\}_{i=1}^k$ up to and including time k. From a Bayesian perspective, the problem is to sequentially compute the filtering pdf $p(\mathbf{x}_k|\mathbf{Z}_k)$, which is given by

$$p(\mathbf{x}_k|\mathbf{Z}_k) = \frac{p(\mathbf{z}_k|\mathbf{x}_k)p(\mathbf{x}_k|\mathbf{Z}_{k-1})}{p(\mathbf{z}_k|\mathbf{Z}_{k-1})}, \tag{2a}$$

$$p(\mathbf{z}_k|\mathbf{Z}_{k-1}) = \int p(\mathbf{z}_k|\mathbf{x}_k)p(\mathbf{x}_k|\mathbf{Z}_{k-1}) \, \mathrm{d}\mathbf{x}_k, \tag{2b}$$

where $p(\mathbf{x}_k|\mathbf{Z}_{k-1})$ is provided by the following time update stage:

$$p(\mathbf{x}_k|\mathbf{Z}_{k-1}) = \int p(\mathbf{x}_k|\mathbf{x}_{k-1})p(\mathbf{x}_{k-1}|\mathbf{Z}_{k-1}) \, \mathrm{d}\mathbf{x}_{k-1}. \tag{2c}$$

The above recursions are initiated by $p(\mathbf{x}_0|\mathbf{Z}_0) = p(\mathbf{x}_0)$ [1]. It is well known that the nonlinear recursive filtering problem only allows analytical solutions in a few special cases, e.g, for linear Gaussian models, where the Kalman filter provides the optimal solution [2]. However, for the general model (1), an analytical solution to the above recursions is intractable and, thus, approximations are needed.

In recent years, sequential Monte Carlo methods, commonly referred to as particle filters, have become an important class of nonlinear filters that efficiently deal with both nonlinearities and non-Gaussian noise [3]–[7]. Until now, a plethora of particle filters have been developed such as, e.g., the bootstrap particle filter [3], the marginal particle filter [8], the auxiliary particle filter [9] and the variable rate particle filter [10]. However, when the dimension of the state space is high, the computational complexity of these filters becomes prohibitively high. As a result, especially for real-time applications with high sampling rates, these particle filters cannot be used.

If the model (1) contains a linear sub-structure, subject to Gaussian noise, then a method which is known as Rao-Blackwellization can be exploited [11]. This results in a filter commonly referred to as the marginalized particle filter (MPF) or the Rao-Blackwellized particle filter, see e.g. [12]–[15]. The MPF consists of a combined particle filter and Kalman filter, that produce estimates with lower or identical covariance as when the standard particle filter was used. Perhaps most important, the MPF allows us to consider high dimensional state estimation problems within the particle filtering framework. Another appealing advantage of the MPF is that in some cases, the computational burden is significantly reduced as well.

The main contribution of this paper is a nonlinear filter, which combines the strengths of both the marginalized particle filter and the auxiliary particle filter. The resulting filter is referred to as the marginalized auxiliary particle filter (MAPF). Related exploitations of the Rao-Blackwellization idea are provided by the Rao-Blackwellized variable rate particle filter, introduced in [16] and the combination of the Rao-Blackwellized particle filter and the marginal particle filter presented in [17]. In order to illustrate the performance of the MAPF, a multi-sensor aircraft tracking example is investigated.

II. MARGINALIZED AUXILIARY PARTICLE FILTER

The well-known idea behind the auxiliary particle filter is to draw N samples $\{\mathbf{x}_k^{(j)}, i^{(j)}\}_{j=1}^N$ from the joint density $p(\mathbf{x}_k, i|\mathbf{Z}_k)$, where i denotes a discrete index [9]. The index i is then omitted, in order to obtain a sample $\{\mathbf{x}_k^{(j)}\}_{j=1}^N$ that approximates the desired filtering density $p(\mathbf{x}_k|\mathbf{Z}_k)$. Compared to the standard particle filter, the auxiliary particle filter can be interpreted as a look ahead method, which at time $k - 1$ predicts which samples will be in regions of high likelihood at time k. As a result, the cost of sampling particles from regions of very low likelihoods is reduced.

Since its introduction in [9], several improvements were proposed to reduce the variance of the auxiliary particle filter [4], [18]. In the following, the marginalized auxiliary particle filter is derived based on the modified auxiliary particle filter presented in [18]. This algorithm has only one resampling step at each time instance and experimentally outperforms the original two-stage resampling algorithm proposed in [9].

The main idea underlying Rao-Blackwellization is to partition the state vector according to

$$\mathbf{x}_k = \left[\begin{array}{c} \mathbf{x}_k^{\mathrm{n}} \\ \mathbf{x}_k^{\mathrm{l}} \end{array} \right], \qquad (3)$$

where $\mathbf{x}_k^{\mathrm{n}}$ denotes the nonlinear state variable and $\mathbf{x}_k^{\mathrm{l}}$ denotes the state variable with conditionally linear Gaussian dynamics. Now, by straightforward application of Bayes' rule we have,

$$p(\mathbf{x}_k^{\mathrm{n}}, \mathbf{x}_k^{\mathrm{l}}, i | \mathbf{Z}_k) = \underbrace{p(\mathbf{x}_k^{\mathrm{l}} | \mathbf{x}_k^{\mathrm{n}}, i, \mathbf{Z}_k)}_{\mathrm{KF}} \cdot \underbrace{p(\mathbf{x}_k^{\mathrm{n}}, i | \mathbf{Z}_k)}_{\mathrm{APF}}, \qquad (4)$$

where the first density $p(\mathbf{x}_k^{\mathrm{l}} | \mathbf{x}_k^{\mathrm{n}}, i, \mathbf{Z}_k)$ is evaluated analytically using the Kalman filter (KF) and the second density $p(\mathbf{x}_k^{\mathrm{n}}, i | \mathbf{Z}_k)$ is approximated using the auxiliary particle filter (APF). In order to exploit the idea of Rao-Blackwellization in the auxiliary particle filter, the following conditional linear Gaussian model is introduced,

$$\mathbf{x}_k^{\mathrm{n}} = \mathbf{f}_{k-1}^{\mathrm{n}}(\mathbf{x}_{k-1}^{\mathrm{n}}) + \mathbf{F}_{k-1}^{\mathrm{n}}(\mathbf{x}_{k-1}^{\mathrm{n}})\mathbf{x}_{k-1}^{\mathrm{l}} + \mathbf{w}_{k-1}^{\mathrm{n}}, \quad (5\mathrm{a})$$
$$\mathbf{x}_k^{\mathrm{l}} = \mathbf{f}_{k-1}^{\mathrm{l}}(\mathbf{x}_{k-1}^{\mathrm{n}}) + \mathbf{F}_{k-1}^{\mathrm{l}}(\mathbf{x}_{k-1}^{\mathrm{n}})\mathbf{x}_{k-1}^{\mathrm{l}} + \mathbf{w}_{k-1}^{\mathrm{l}}, \quad (5\mathrm{b})$$
$$\mathbf{z}_k = \mathbf{h}_k(\mathbf{x}_k^{\mathrm{n}}) + \mathbf{H}_k(\mathbf{x}_k^{\mathrm{n}})\mathbf{x}_k^{\mathrm{l}} + \mathbf{e}_k, \qquad (5\mathrm{c})$$

where $\mathbf{w}_{k-1}^{\mathrm{n}}$, $\mathbf{w}_{k-1}^{\mathrm{l}}$ and $\mathbf{e}_k$ are assumed to be white and Gaussian distributed according to

$$\left[\begin{array}{c} \mathbf{w}_{k-1}^{\mathrm{n}} \\ \mathbf{w}_{k-1}^{\mathrm{l}} \end{array} \right] \sim \mathcal{N}(\mathbf{0}, \mathrm{diag}(\mathbf{Q}_{k-1}^{\mathrm{n}}, \mathbf{Q}_{k-1}^{\mathrm{l}})), \quad \mathbf{e}_k \sim \mathcal{N}(0, \mathbf{R}_k). \qquad (6)$$

Here, the process noises $\mathbf{w}_{k-1}^{\mathrm{n}}$ and $\mathbf{w}_{k-1}^{\mathrm{l}}$ are assumed to be independent. This is no restriction, since the case of dependent noise can be reduced to the case of independent noise using a Gram-Schmidt procedure [19]. Furthermore, the density of $\mathbf{x}_0^{\mathrm{l}}$ is Gaussian, i.e., $\mathbf{x}_0^{\mathrm{l}} \sim \mathcal{N}(\hat{\mathbf{x}}_0^{\mathrm{l}}, \mathbf{P}_0)$. The density of $\mathbf{x}_0^{\mathrm{n}}$ can be arbitrary, but it is assumed known.

The marginalzed auxiliary particle filter for the case $\mathbf{H}_k(\mathbf{x}_k^{\mathrm{n}}) = \mathbf{0}$ is summarized in Table I. For the sake of notational brevity, the dependence of $\mathbf{x}_{k-1}^{\mathrm{n},(i)}$ in $\mathbf{f}_{k-1}^{\mathrm{n}}$, $\mathbf{f}_{k-1}^{\mathrm{l}}$, $\mathbf{F}_{k-1}^{\mathrm{n}}$ and $\mathbf{F}_{k-1}^{\mathrm{l}}$ is denoted as $\mathbf{f}_{k-1}^{\mathrm{n},(i)}$, $\mathbf{f}_{k-1}^{\mathrm{l},(i)}$, $\mathbf{F}_{k-1}^{\mathrm{n},(i)}$ and $\mathbf{F}_{k-1}^{\mathrm{l},(i)}$ below.

Let us now briefly explain the MAPF. After the initialization stage, the so-called first stage weights are evaluated (step (2) in Table I). These weights are given by the importance density $\pi(i|\mathbf{Z}_k)$ which is proportional to the predictive likelihood. The subsequent resampling stage assures that only particles with high predictive likelihood will be used.

The time update stage which follows can be split into a PF and KF time update step (step (4) in Table I). In the PF time update, a prediction of the nonlinear states $\mathbf{x}_k^{\mathrm{n}}$ is obtained from the importance density $\pi(\mathbf{x}_k^{\mathrm{n}}|i^{(j)}, \mathbf{Z}_k)$. Given the nonlinear states, the model in (5) is conditional linear. The conditioning implies

TABLE I
MARGINALIZED AUXILIARY PARTICLE FILTER

(1) Initialization:
  – For $i = 1, ..., N$, initialize the particles $\mathbf{x}_0^{\mathrm{n},(i)} \sim p(\mathbf{x}_0^{\mathrm{n}})$ and weights $w_0^{(i)} = \frac{1}{N}$ and set $\{\mathbf{x}_{0|0}^{\mathrm{l},(i)}, \mathbf{P}_{0|0}^{(i)}\} = \{\hat{\mathbf{x}}_0^{\mathrm{l}}, \mathbf{P}_0\}$.

(2) Time Update and Measurement Update (First stage weights), $i = 1, ..., N$:
  – Determine $\boldsymbol{\mu}_k^{\mathrm{n},(i)}$ from $\mathcal{N}(\mathbf{x}_k^{\mathrm{n}}; \bar{\mathbf{x}}_{k|k-1}^{\mathrm{n},(i)}, \bar{\mathbf{P}}_{k|k-1}^{\mathrm{n},(i)})$, e.g., take the mean $\boldsymbol{\mu}_k^{\mathrm{n},(i)} = \bar{\mathbf{x}}_{k|k-1}^{\mathrm{n},(i)}$, where

  $$\bar{\mathbf{x}}_{k|k-1}^{\mathrm{n},(i)} = \mathbf{f}_{k-1}^{\mathrm{n},(i)} + \mathbf{F}_{k-1}^{\mathrm{n},(i)} \mathbf{x}_{k-1|k-1}^{\mathrm{l},(i)},$$
  $$\bar{\mathbf{P}}_{k|k-1}^{\mathrm{n},(i)} = \mathbf{F}_{k-1}^{\mathrm{n},(i)} \mathbf{P}_{k-1|k-1}^{(i)} (\mathbf{F}_{k-1}^{\mathrm{n},(i)})^{\mathsf{T}} + \mathbf{Q}_{k-1}^{\mathrm{n}}.$$

  – Evaluate the first stage weights $\tilde{w}_k^{(i)} = \pi(i|\mathbf{Z}_k) \propto w_{k-1}^{(i)} \mathcal{N}(\mathbf{z}_k; \tilde{\mathbf{z}}_k^{(i)}, \tilde{\mathbf{S}}_k^{(i)})$, where $\tilde{\mathbf{z}}_k^{(i)} = \mathbf{h}_k(\boldsymbol{\mu}_k^{\mathrm{n},(i)})$ and $\tilde{\mathbf{S}}_k^{(i)} = \mathbf{R}_k$ and normalize the weights according to $w_k^{(i)} = \tilde{w}_k^{(i)} / \sum_{m=1}^{N} \tilde{w}_k^{(m)}$.

(3) Resampling:
  – Perform systematic resampling [5] and store for each resampled particle the parent index, denoted by $i^{(j)}$.

(4) Time Update, $j = 1, ..., N$:
  – PF: Draw samples $\mathbf{x}_k^{\mathrm{n},(j)} \sim \pi(\mathbf{x}_k^{\mathrm{n}}|i^{(j)}, \mathbf{Z}_k) = \mathcal{N}(\mathbf{x}_k^{\mathrm{n}}; \bar{\mathbf{x}}_{k|k-1}^{\mathrm{n},(i^j)}, \bar{\mathbf{P}}_{k|k-1}^{\mathrm{n},(i^j)})$.
  – KF: Evaluate

  $$\mathbf{x}_{k|k-1}^{\mathrm{l},(j)} = \mathbf{f}_{k-1}^{\mathrm{l},(i^j)} + \mathbf{F}_{k-1}^{\mathrm{l},(i^j)} \mathbf{x}_{k-1|k-1}^{\mathrm{l},(i^j)} + \mathbf{L}_{k-1}^{(i^j)} (\mathbf{z}_{k-1}^{(j)} - \mathbf{F}_{k-1}^{\mathrm{n},(i^j)} \mathbf{x}_{k-1|k-1}^{\mathrm{l},(i^j)}),$$
  $$\mathbf{P}_{k|k-1}^{(j)} = \mathbf{F}_{k-1}^{\mathrm{l},(i^j)} \mathbf{P}_{k-1|k-1}^{(i^j)} (\mathbf{F}_{k-1}^{\mathrm{l},(i^j)})^{\mathsf{T}} + \mathbf{Q}_{k-1}^{\mathrm{l}} - \mathbf{L}_{k-1}^{(j)} \mathbf{N}_{k-1}^{(j)} (\mathbf{L}_{k-1}^{(j)})^{\mathsf{T}},$$

  where

  $$\mathbf{N}_{k-1}^{(j)} = \mathbf{F}_{k-1}^{\mathrm{n},(i^j)} \mathbf{P}_{k-1|k-1}^{(i^j)} (\mathbf{F}_{k-1}^{\mathrm{n},(i^j)})^{\mathsf{T}} + \mathbf{Q}_{k-1}^{\mathrm{n}},$$
  $$\mathbf{L}_{k-1}^{(j)} = \mathbf{F}_{k-1}^{\mathrm{l},(i^j)} \mathbf{P}_{k-1|k-1}^{(i^j)} (\mathbf{F}_{k-1}^{\mathrm{n},(i^j)})^{\mathsf{T}} (\mathbf{N}_{k-1}^{(j)})^{-1},$$
  $$\mathbf{z}_{k-1}^{(j)} = \mathbf{x}_k^{\mathrm{n},(j)} - \mathbf{f}_{k-1}^{\mathrm{n},(i^j)}.$$

(5) Measurement Update (Second stage weights), $j = 1, ..., N$:
  – PF: Evaluate the second stage weights $\tilde{w}_k^{(j)} = \frac{\mathcal{N}(\mathbf{z}_k; \hat{\mathbf{z}}_k^{(j)}, \mathbf{S}_k^{(j)})}{\mathcal{N}(\mathbf{z}_k; \tilde{\mathbf{z}}_k^{(i^j)}, \tilde{\mathbf{S}}_k^{(i^j)})}$, with $\hat{\mathbf{z}}_k^{(j)} = \mathbf{h}_k(\mathbf{x}_k^{\mathrm{n},(j)})$ and $\mathbf{S}_k^{(j)} = \mathbf{R}_k$, and normalize the weights according to $w_k^{(j)} = \tilde{w}_k^{(j)} / \sum_{m=1}^{N} \tilde{w}_k^{(m)}$.
  – KF: Set $\mathbf{x}_{k|k}^{\mathrm{l},(j)} = \mathbf{x}_{k|k-1}^{\mathrm{l},(j)}$ and $\mathbf{P}_{k|k}^{(j)} = \mathbf{P}_{k|k-1}^{(j)}$.

(6) Set $i := j$ and $k := k+1$ and iterate from step (2).

that (5a) can be interpreted as a measurement equation with the artificial measurement $\mathbf{z}_{k-1}^{(j)}$. Thus, a measurement update together with a time update step is performed for each particle in the KF time update step, yielding the estimates of the linear states $\mathbf{x}_{k|k-1}^{1,(j)}$ and the corresponding error covariances $\mathbf{P}_{k|k-1}^{(j)}$. The measurement update step can also be decomposed into a PF and KF measurement update stage (step (5) in Table I). In the PF measurement update stage, the second stage weights are evaluated taking into account the actual measurements $\mathbf{z}_k$. In the KF measurement update, the measurements $\mathbf{z}_k$, cf. (5c), provide no new information about the linear states $\mathbf{x}_k^{l}$, since $\mathbf{H}_k(\mathbf{x}_k^{n}) = \mathbf{0}$. Hence, the measurement equation (5c) cannot be used in estimating the linear states.

An important special case for the marginalized auxiliary particle filter arises when the matrices $\mathbf{F}_{k-1}^{n}$, $\mathbf{F}_{k-1}^{l}$ and $\mathbf{H}_k$ are independent of $\mathbf{x}_k^{n}$. Then,

$$\mathbf{P}_{k|k}^{(j)} = \mathbf{P}_{k|k}, \quad \forall j = 1,...N, \qquad (7)$$

which implies that only 1 instead of N Riccati recursions is needed. As a result, the computational complexity can be significantly reduced, which of course is useful, especially for real-time implementations.

III. Illustrating Example

The performance of the proposed marginalized auxiliary particle filter is investigated using a well-known radar target tracking problem. It is assumed that an aircraft is moving in the 2-D plane, which is modelled according to

$$\mathbf{x}_k = \begin{bmatrix} \mathbf{I} & T \cdot \mathbf{I} & T^2/2 \cdot \mathbf{I} \\ \mathbf{0} & \mathbf{I} & T \cdot \mathbf{I} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix} \cdot \mathbf{x}_{k-1} + \mathbf{w}_{k-1}, \quad (8a)$$

$$\mathbf{z}_k = \begin{bmatrix} \sqrt{p_x^2 + p_y^2} \\ \arctan(p_y/p_x) \end{bmatrix} + \mathbf{e}_k, \qquad (8b)$$

where the state vector $\mathbf{x}_k$ contains the position, the velocity and the acceleration of the aircraft, i.e. $\mathbf{x}_k = [p_x, p_y, v_x, v_y, a_x, a_y]^\mathsf{T}$, and $\mathbf{I}$ denotes the identity matrix of size 2. A stationary radar sensor at position $\mathbf{x}_{\text{sens}} = [0,0]^\mathsf{T}$ records range and azimuth measurements, denoted $\mathbf{z}_k$, of the aircraft using a sampling time of $T = 1$ seconds. The process and measurement noise $\mathbf{w}_{k-1}$ and $\mathbf{e}_k$ are assumed to be zero-mean white Gaussian noise sequences with covariance matrices $\mathbf{Q}_{k-1} = \text{Cov}[\mathbf{w}_{k-1}] = \text{diag}[4, 4, 4, 4, 0.01, 0.01]$ and $\mathbf{R}_k = \text{Cov}[\mathbf{e}_k] = \text{diag}[100, 10^{-6}]$. Since the model in (8) is conditional linear Gaussian, the marginalized auxiliary particle filter can be readily applied.

The proposed marginalized auxiliary particle filter is investigated through simulations based on $N_{\text{mc}} = 1000$ Monte Carlo trials. For each trial, the aircraft's trajectory was generated from (8a), where the initial state vector $\mathbf{x}_0$ was drawn randomly from a Gaussian distribution with mean $\hat{\mathbf{x}}_0 = [2000, 2000, 20, 20, 0, 0]^\mathsf{T}$ and error covariance matrix $\mathbf{P}_0 = \text{diag}[4, 4, 16, 16, 0.04, 0.04]$.

The performance of the MAPF is compared to the modified auxiliary particle filter (APF) [18], the marginalized particle filter (MPF) [15] with transitional prior as importance density and the posterior Cramér-Rao lower bound (PCRLB) [20]. Here, it is worth noting that in addition simulations have been carried out for the standard (bootstrap) particle filter. However, the achieved performance results were even worse than for the APF and thus are not shown.

In Figs. 1 and 2, the root mean squared error (RMSE) and PCRLB of the aircraft's position and acceleration are shown for the different particle filter algorithms using $N = 250$ particles. From Fig. 1 it can be seen that in terms of position RMSE the proposed MAPF yields the best results, followed by the MPF and the APF. In terms of acceleration RMSE, cf. Fig. 2, the APF yields the worst results, whereas the performance of the MPF and MAPF are practically the same as both filters attain the PCRLB.

In addition, simulations were performed for the different filters with $N = 100$, 250 and 2000 particles. The results in terms of the number of diverged trials, average RMSE and processing time are summarized in Table II. Here, the processing time denotes the time it takes to perform a single iteration of the algorithm and serves here as an indicator for the expected computational complexity.

The simulations show that compared to the APF, the MAPF provides the best performance with approximately the same computational complexity. As expected, the computational complexity of MAPF is larger than that of the MPF. However, the performance of the MAPF is superior to the performance of the MPF. This becomes obvious especially when N is small, whereas for large N the performance is approximately the same.

IV. Conclusion

In this paper, we have proposed the marginalized auxiliary particle filter for nonlinear systems with a conditionally linear Gaussian sub-structure. The filter has been applied to a multi-sensor aircraft tracking problem and its performance is compared with the auxiliary particle filter and the marginalized particle filter. Simulation results have shown that the marginalized auxiliary particle filter outperforms the auxiliary particle filter in terms of RMSE, while the computational complexity is almost equal. Compared to the marginalized particle filter, the strength of the marginalized auxiliary particle filter is its superior performance when the number of particles is small. However, the price that has to be paid is an increased computational complexity.

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TABLE II
NUMBER OF DIVERGED TRIALS, AVERAGE RMSE AND PROCESSING TIME FOR THE AUXILIARY PARTICLE FILTER (APF), MARGINALIZED PARTICLE FILTER (MPF) AND MARGINALIZED AUXILIARY PARTICLE FILTER (MAPF)

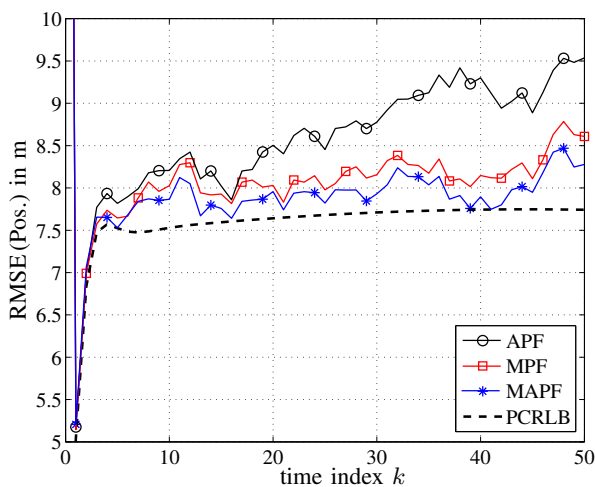| Filter	APF			MPF			MAPF		
N	100	250	2000	100	250	2000	100	250	2000
Diverged	12	0	0	0	0	0	0	0	0
Pos. in m	9.99	8.58	7.79	8.65	8.04	7.75	8.06	7.86	7.73
Vel. in m/s	5.46	5.24	5.03	5.16	5.05	4.99	5.07	5.02	4.98
Acc. in m/s^2	0.82	0.74	0.63	0.59	0.59	0.59	0.59	0.59	0.59
Time in s	0.07	0.17	1.36	0.03	0.07	0.50	0.07	0.18	1.38

Fig. 1.   Position RMSE for the different particle filter algorithms for $N = 250$ and the corresponding PCRLB.
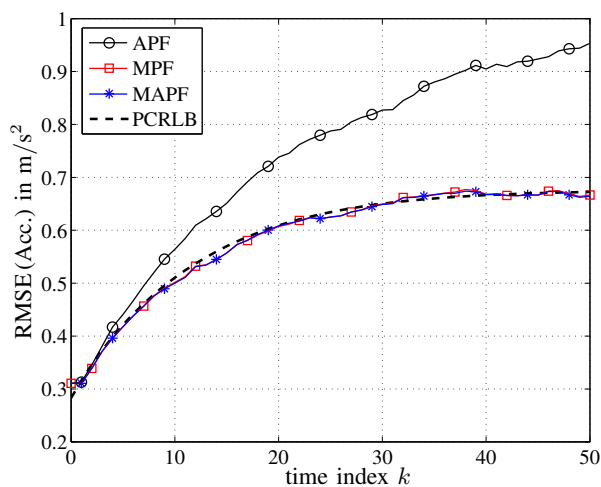

Fig. 2.   Acceleration RMSE for the different particle filter algorithms for $N = 250$ and the corresponding PCRLB.

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