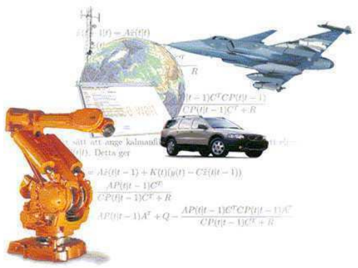
# Robust Multivariable Control

*Lecture 8*



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# Today's topics

- Uncertainty descriptions
- LFTs
- Robustness for stability
- Robustness for performance
- $D$ scalings
- Convexity

# Model uncertainties

Previously we have used descriptions of uncertainties that are not very detailed.

- Phase and amplitude margins
- Coprime factorizations, $\nu$-gap

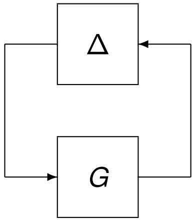
We will now see how we can specify uncertainties in a more structured way.

Robust control has to cope with systems with uncertainties.

How can we specify uncertainties and how can we analyze such systems?

# Small Gain Theorem

For analysis the small gain theorem is an important tool.

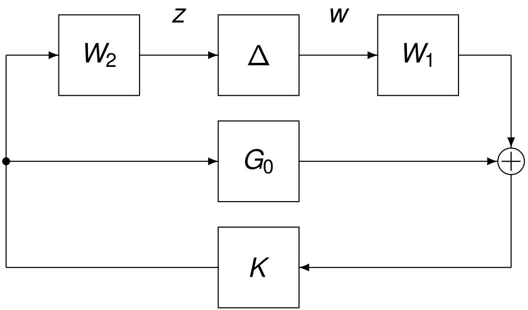



$G$ and $\Delta$ are stable.

$$\left\{ \begin{array}{l} \|\Delta\|_\infty \leq 1/\gamma \\ \|G\|_\infty < \gamma \end{array} \right. \Rightarrow \text{closed loop system stable.}$$

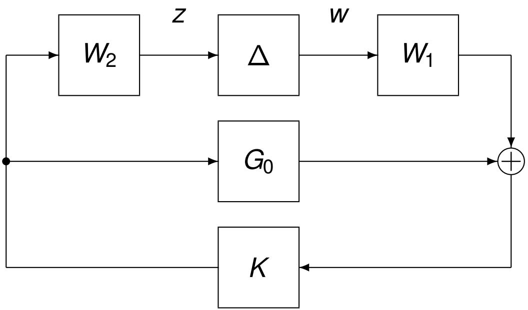
$\Delta$ can also be a nonlinear system with limited power gain (induced $L_2$-norm).

# Additive uncertainties




For which uncertainties, $\Delta$, is the closed loop system stable. The nominal system with $\Delta = 0$ is assumed to be stable.

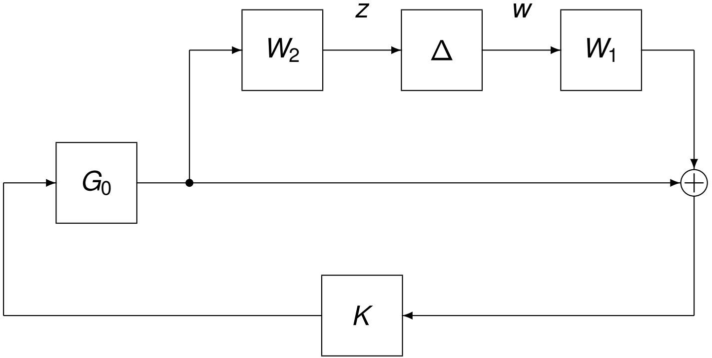
# Additive uncertainties




The transfer function from $w$ to $z$: $W_2 K \underbrace{(I - G_0 K)^{-1}}_{S_0} W_1 = W_2 K S_0 W_1$

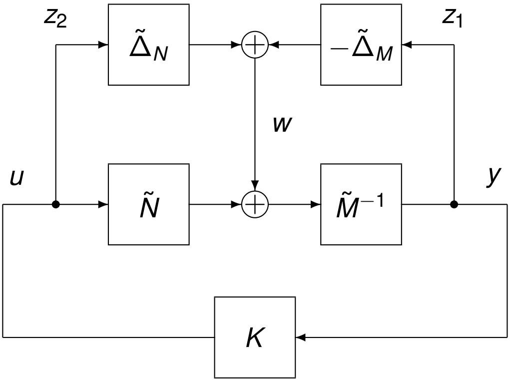
Using the small gain theorem, $\|\Delta\|_\infty \leq 1/\gamma$ and $\|W_2 K S_0 W_1\|_\infty < \gamma$ imply stability.

# Multiplicative uncertainties




Using the small gain theorem, $\|\Delta\|_\infty \leq 1/\gamma$ and
$\|W_2 \underbrace{G_0 K (I - G_0 K)^{-1}}_{T_0} W_1\|_\infty = \|W_2 T_0 W_1\|_\infty < \gamma$ imply stability.

# Uncertainty descriptions with coprime factorizations



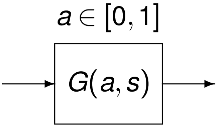

The transfer function from $w$ to $\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$:

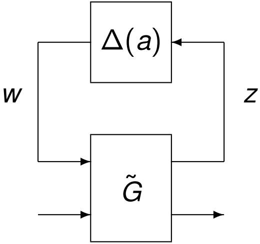
$$\begin{bmatrix} I \\ K \end{bmatrix} (I - G_0 K)^{-1} \tilde{M}^{-1}$$

Using the small gain theorem, $\left\| \begin{bmatrix} \tilde{\Delta}_M & \tilde{\Delta}_N \end{bmatrix} \right\|_\infty \leq 1/\gamma$ and $\left\| \begin{bmatrix} I \\ K \end{bmatrix} (I - G_0 K)^{-1} \tilde{M}^{-1} \right\|_\infty < \gamma$ imply stability.

# Parametric uncertainties

Often a system can include parameters that are uncertain, but within given bounds, for instance:



# An example

$$G(s) = \frac{1}{s^2 - a}, \qquad a \in [0, 1]$$

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ a & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$a = a_0 + a_1\delta = \frac{1}{2} + \frac{1}{2}\delta, \qquad \delta \in [-1, 1]$$

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ a & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$= \begin{bmatrix} 0 & 1 \\ a_0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ a_1 \end{bmatrix} \overbrace{\delta \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{z}}^{w=\delta z} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$= \begin{bmatrix} 0 & 1 \\ a_0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ a_1 \end{bmatrix} w + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$z = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ a_0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ a_1 \end{bmatrix} w + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

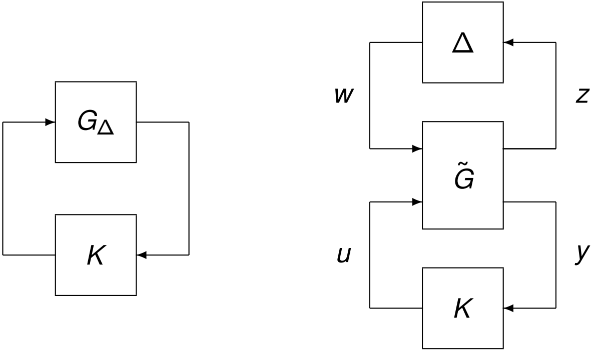
$$z = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$w = \delta z$$

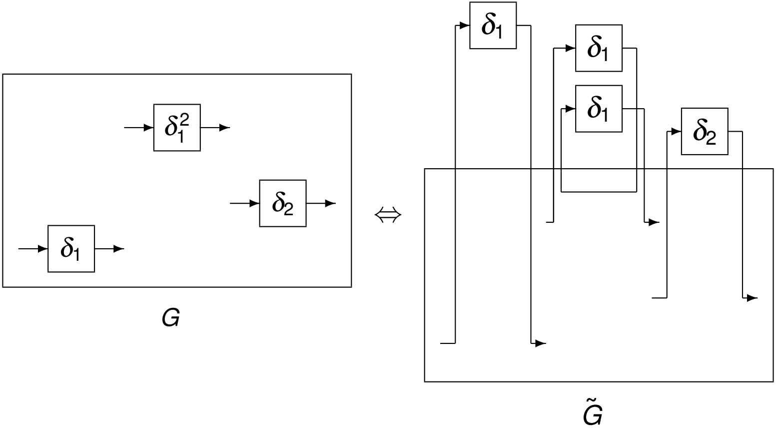
$$\tilde{G} = \left[\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ a_0 & 0 & a_1 & 1 \\ \hline 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array}\right]$$

# Robustness for stability




Find a $K$ such that $\|\mathscr{F}_\ell(\tilde{G}, K)\|_\infty < \gamma$. Then the system is stable for all $\|\Delta\| \leq 1/\gamma$. Here we use the notation $\mathscr{F}_l(\tilde{G}, K)$ to denote the system $\tilde{G}$ with $K$ as a feedback.

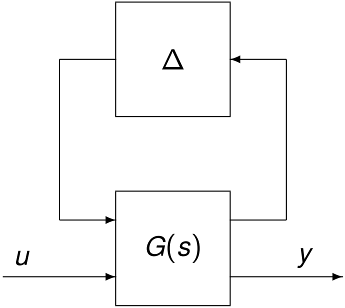
# Pull out the parameters




Here we get repeated $\delta_1$ parameters.

# LFT representations

$$\Delta \star G(s) = \mathscr{F}_u(G(s), \Delta)$$




$$\Delta = \begin{bmatrix} \delta_1 & 0 & 0 & 0 \\ 0 & \delta_1 & 0 & 0 \\ 0 & 0 & \delta_1 & 0 \\ 0 & 0 & 0 & \delta_2 \end{bmatrix}$$

# Star product

Upper LFT:

$$\mathscr{F}_u(M,\Delta) = M_{22} + M_{21}\Delta(I - M_{11}\Delta)^{-1}M_{12}$$

Lower LFT:

$$\mathscr{F}_\ell(M,\Delta) = M_{11} + M_{12}\Delta(I - M_{22}\Delta)^{-1}M_{21}$$

Redheffer's star product:

$$Q \star M = \begin{bmatrix} \mathscr{F}_\ell(Q, M_{11}) & Q_{12}(1 - M_{11}Q_{22})^{-1}M_{12} \\ M_{21}(I - Q_{22}M_{11})^{-1}Q_{21} & \mathscr{F}_u(M, Q_{22}) \end{bmatrix}$$

# General linear relations

If $M = M_0 + \delta M_1$ we can use singular value decomposition of $M_1 = U\Sigma V^T$ where $\Sigma$ is of rank $r$.

$$M = M_0 + \delta U\Sigma V^T = \mathscr{F}_u\left(\begin{bmatrix} 0 & V^T \\ U\Sigma & M_0 \end{bmatrix}, \delta I_r\right)$$

# Rational functions

Rational functions can be described by LFTs.

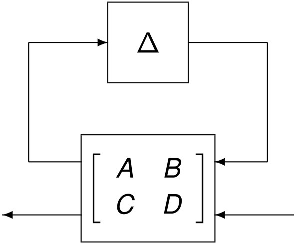
For instance,

$$G(\delta_1, \delta_2) = \frac{\delta_1 \delta_2}{1 + \delta_1 + \delta_2},$$

which can be written as

$$G(\delta_1, \delta_2) = \delta_2 \star \left[\begin{array}{c|c} -\frac{1}{1+\delta_1} & 1 \\ \hline \frac{\delta_1}{1+\delta_1} & 0 \end{array}\right] = \begin{bmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{bmatrix} \star \left[\begin{array}{cc|c} -1 & 1 & 0 \\ 1 & -1 & 1 \\ \hline 1 & 0 & 0 \end{array}\right]$$

# General case




$$\mathscr{F}_u\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}, \Delta\right) = D + C\Delta(I - A\Delta)^{-1}B$$
$$= D + C(I - \Delta A)^{-1}\Delta B$$

# Continuous time and discrete time systems

$$\mathscr{F}_u\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}, \Delta\right) = D + C(I - \Delta A)^{-1}\Delta B$$

Compare with continuous-time systems ($\Delta = s^{-1}I$):

$$\mathscr{F}_u\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}, s^{-1}I\right) = D + C(I - s^{-1}A)^{-1}s^{-1}B = D + C(sI - A)^{-1}B = G(s)$$

Compare with discrete-time systems ($\Delta = z^{-1}I$):

$$\mathscr{F}_u\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}, z^{-1}I\right) = D + C(I - z^{-1}A)^{-1}z^{-1}B = D + C(zI - A)^{-1}B = G(z)$$

The similarity between dynamic and LFT models allows us to apply several well-known tools on LFT models, for instance model reduction.

# The LFT model is not unique

For instance consider
$G(\delta_1, \delta_2) = \begin{bmatrix} \delta_1 & \delta_2 & \delta_1\delta_2 \end{bmatrix} = \mathscr{F}_u(M, \Delta) = \Delta \star M$, which can be represented as an LFT with

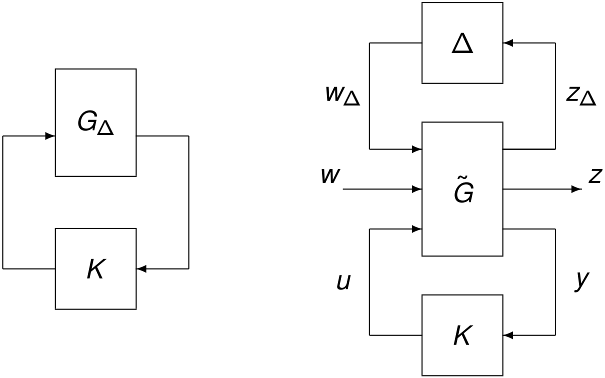
$$M = \left[\begin{array}{ccc|ccc} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 1 & 1 & 0 & 0 & 0 \end{array}\right] \qquad \Delta = \begin{bmatrix} \delta_1 & 0 & 0 \\ 0 & \delta_1 & 0 \\ 0 & 0 & \delta_2 \end{bmatrix}$$

or

$$M = \left[\begin{array}{ccc|ccc} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ \hline 1 & 0 & 1 & 0 & 0 & 0 \end{array}\right] \qquad \Delta = \begin{bmatrix} \delta_2 & 0 & 0 \\ 0 & \delta_2 & 0 \\ 0 & 0 & \delta_1 \end{bmatrix}$$

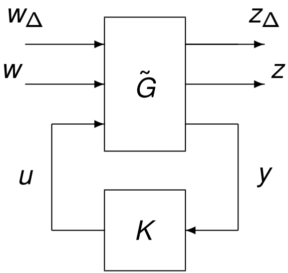
depending on the order of the parameter extraction.

# Robustness for performance




Find a controller that gives robustness with respect to performance, that is to say $H_\infty$-norm $< \gamma$ for all possible systems where $\|\Delta\|_\infty \leq 1$.

# Robustness for performance




Minimize $H_\infty$-gain from $\begin{bmatrix} w_\Delta \\ w \end{bmatrix}$ to $\begin{bmatrix} z_\Delta \\ z \end{bmatrix}$.

# Performance requirements

If $\gamma < 1$ the the performance requirements are satisfied since

$$\left\| \begin{bmatrix} z_\Delta \\ z \end{bmatrix} \right\|_2^2 \leq \gamma^2 \left\| \begin{bmatrix} w_\Delta \\ w \end{bmatrix} \right\|_2^2$$

$$\|z\|_2^2 + \|z_\Delta\|_2^2 \leq \gamma^2 \|w\|_2^2 + \gamma^2 \|w_\Delta\|_2^2$$

$$\|z\|_2^2 \leq \gamma^2 \|w\|_2^2$$

since $\|\Delta\| \leq 1$ implies $\|w_\Delta\|_2^2 \leq \|z_\Delta\|_2^2$.

Consequently the performance requirements are satisfied for all $\|\Delta\| \leq 1$.

It is also possible to show that the performance requirements are satisfied for all $\|\Delta\| \leq 1/\gamma$, since $\|w_\Delta\|_2^2 \leq \gamma^{-2} \|z_\Delta\|_2^2$.

# Conservatism

A problem with this type of analysis is that it sometimes becomes conservative. For instance:

$$G_1(j\omega) = \tfrac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad \|G(j\omega)\| = 1$$

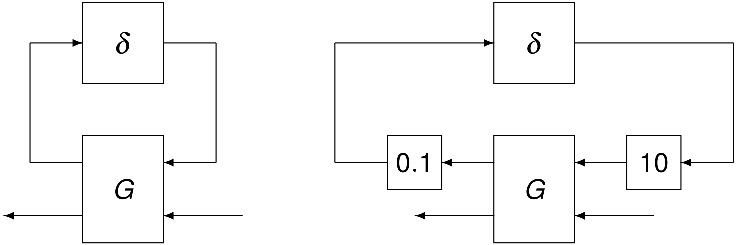
but

$$G_2(j\omega) = \tfrac{1}{2}\begin{bmatrix} 1 & 10 \\ 0.1 & 1 \end{bmatrix}, \quad \|G(j\omega)\| = 5.05$$

These are equivalent if we use $\delta$ as feedback:

$$\mathscr{F}_u(G_1, \delta) = \mathscr{F}_u(G_2, \delta)$$

# Scalings

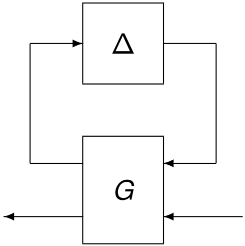


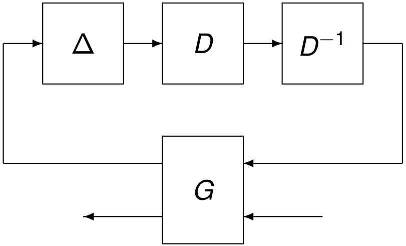

This depends on how the uncertainty model has been built.

It depends on the scalings!

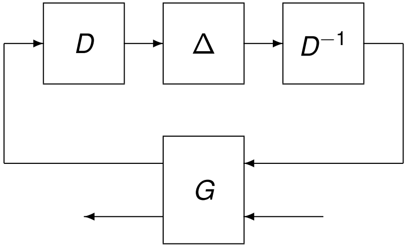
Compare this with similarity transformations.

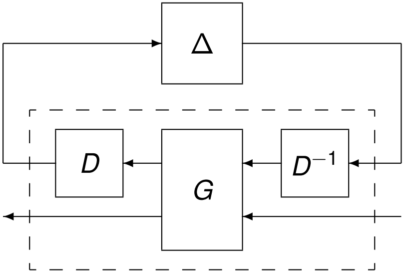
# $D$ scalings: $D\Delta = \Delta D$, $D$ invertible



# *D* scalings



# $D$ scalings

Allow more freedom in the analysis by including invertible scalings, $D$. The scalings and the uncertainties, $\Delta$, must have structures that commute.

$$D\Delta = \Delta D$$

For instance if

$$\Delta = \begin{bmatrix} \delta_1 & 0 & 0 & 0 \\ 0 & \delta_1 & 0 & 0 \\ 0 & 0 & \delta_1 & 0 \\ 0 & 0 & 0 & \delta_2 \end{bmatrix}$$

then

$$D = \begin{bmatrix} d_{11} & d_{12} & d_{13} & 0 \\ d_{21} & d_{22} & d_{23} & 0 \\ d_{31} & d_{32} & d_{33} & 0 \\ 0 & 0 & 0 & d_{44} \end{bmatrix}$$

commutes with $\Delta$.

# $D$ scalings

Since $\Delta = D^{-1}D\Delta = D^{-1}\Delta D$, we can improve the result from the small gain theorem

$$\|DGD^{-1}\|_\infty < 1.$$

If the parameters are time-varying, then $D$ must be constant.
If the parameters are constant or time-invariant systems, then we can allow for dynamic scalings:

$$\|D(s)G(s)D^{-1}(s)\|_\infty < 1.$$

In this case we can perform a frequency sweep and minimize

$$\left\|D_\omega G(j\omega)D_\omega^{-1}\right\| < 1$$

with respect for $D_\omega$ for each frequency, $\omega$.

# A convex problem

We will now regard the problem of minimizing $\|DGD^{-1}\|$ with respect to a non-singular $D$.

We know that $\|U\| < \gamma$ is equivalent to $\bar{\lambda}(U^*U) < \gamma^2$, which we also can express as $U^*U - \gamma^2 I \prec 0$ (negative definite):

$$\|DGD^{-1}\| < \gamma$$
$$\left(DGD^{-1}\right)^* \left(DGD^{-1}\right) - \gamma^2 I \prec 0$$
$$D^{-*}G^*D^*DGD^{-1} - \gamma^2 I \prec 0$$
$$G^*D^*DG - \gamma^2 D^*D \prec 0$$

Introduce $P = P^* = D^*D \succ 0$ (positive definite):

$$G^*PG - \gamma^2 P \prec 0$$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# LMIs

$$G^* P G - \gamma^2 P \prec 0, \quad P \succ 0$$

This is a linear matrix inequality (LMI).

This is a convex problem, since if $P_1$ and $P_2$ are two solutions, then also $\lambda P_1 + (1-\lambda)P_2$ is a solution if $\lambda \in [0,1]$, since

$$\begin{aligned} G^* P G - \gamma^2 P &= G^*(\lambda P_1 + (1-\lambda)P_2)G - \gamma^2(\lambda P_1 + (1-\lambda)P_2) = \\ &= \lambda \underbrace{(G^* P_1 G - \gamma^2 P_1)}_{\prec 0} + (1-\lambda)\underbrace{(G^* P_2 G - \gamma^2 P_2)}_{\prec 0} \prec 0 \end{aligned}$$

It is relatively easy to find a global solution to a convex problem, since the local solution is also the global minimum.