Discretizing stochastic dynamical systems using Lyapunov equations

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Abstract: Stochastic dynamical systems are fundamental in state estimation, system identification and control. System models are often provided in continuous time, while a major part of the applied theory is developed for discrete-time systems. Discretization of continuous-time models is hence fundamental. We present a novel algorithm using a combination of Lyapunov equations and analytical solutions, enabling efficient implementation in software. The proposed method circumvents numerical problems exhibited by standard algorithms in the literature. Both theoretical and simulation results are provided.

1. INTRODUCTION

Dynamical processes in engineering and physics have for a long time successfully been modeled with continuous-time differential equations. In order to account for uncertainties, these equations are usually driven by an unknown stochastic process called process noise. This noise is ideally modeled as completely "white" in order to obtain the Markov property, which is required in recursive Bayesian inference, such as Kalman filtering. However, in order to implement such filtering, the continuous-time model has to be discretized. Such discretization includes solving an integral involving the matrix exponential on the form

$$Q_{T_k} = \int_0^{T_k} e^{A\tau} S e^{A^\mathsf{T}\tau} d\tau, \tag{1}$$

where A is the system matrix and S the process noise covariance matrix for the time-continuous system [1] , and where where Q_{T_k} is the process noise covariance for the the discrete-time system

$$\dot{\boldsymbol{x}}(t) = A\boldsymbol{x}(t) + \boldsymbol{w}(t), \quad E[\boldsymbol{w}(t)\boldsymbol{w}(\tau)^\mathsf{T}] = S\delta(t-\tau) \tag{2a}$$

$$\boldsymbol{x}_{k+1} = F_{T_k}\boldsymbol{x}_k + \boldsymbol{w}_k, \quad E[\boldsymbol{w}_k\boldsymbol{w}_l^\mathsf{T}] = Q_{T_k}\delta_{kl}. \tag{2b}$$

Here, $T_k = t_{k+1} - t_k$ denotes the sampling time.

We propose an algorithm for solving (1) by decomposing the problem into subproblems and then solve these subproblems either analytically or using a combination of Lyapunov and Sylvester equations.

In many practical applications the discrete-time process noise covariance is modeled and tuned directly, rather than discretized from its continuous-time counterpart. However, in certain scenarios the dependency between the discrete-time process noise covariance and the sampling time is important. If the filtering should work on different devices with different sampling frequencies, this dependency should be properly modeled to guarantee the same dynamical behavior of the filter. Further, in data with non-equidistant sampling the process noise covariance has to be rescaled at each time instant.

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[1] Since $\boldsymbol{w}(t)$ is not square Riemann integrable, the model (2a) does not have any mathematical meaning (Jazwinski, 1970). However, we can still intuitively think of it as a stochastic differential equation driven by white noise.

In the literature there exist different algorithms for computing the integral (1). The probably most well-cited one was presented by Van Loan (1978), which involves computing the matrix exponential for an augmented $2n \times 2n$ matrix followed by a matrix multiplication of two resulting submatrices. This method does not require any assumption on the model, however the resulting matrix becomes ill-conditioned if the sampling time is large or if the poles of the system are fast.

In this work we present an alternative method for solving (1). This method is based on a Lyapunov equation which characterizes the solution of (1). However, since Lyapunov equations cannot be solved if the system contains integrators (Antoulas, 2005), the problem is decomposed into subproblems where the integrators are treated separately. As will be explained, one set of subproblems cannot be solved using Lyapunov equations, but they do have an analytical solution of (1). Conversely, the remaining set of subproblems do not have a closed form solution of (1), but then the method with Lyapunov equations works fine. The algorithm involves computing the matrix exponential of the $n \times n$ system matrix rather than an augmented $2n \times 2n$ matrix as required by the solution by Van Loan. Furthermore, the proposed algorithm circumvents some numerical problems in the method proposed by Van Loan.

An extended version of this work has been accepted for IFAC world congress 2014 and is also available online (Wahlström et al., 2014)

2. DISCRETIZATION BY LYAPUNOV EQUATIONS

It is trivial to realize that the discrete-time system matrix F_{T_k} equals the matrix exponential expression

$$F_{T_k} = e^{AT_k} \tag{3a}$$

which is achieved by integrating (2a) from t_k to t_{k+1}. However, it is not as trivial to find the discrete-time process noise covariance Q_{T_k}, which requires to find a solution to the integral (1), (Jazwinski, 1970). We propose a solution based on solving the following Lyapunov equation

$$AQ_{T_k} + Q_{T_k}A^\mathsf{T} = -V_{T_k}, \qquad V_{T_k} = S - F_{T_k}SF_{T_k}^\mathsf{T} \tag{3b}$$

This solution is similar to the one presented by Axelsson and Gustafsson (2012) derived from a continuous-time differential Lyapunov equation. It can indeed be proven that (1) satisfies the Lyapunov equation (3b), for proof and more details see Wahlström et al. (2014).

However, (3) has not a unique solution if and only if A and $-A$ have any common eigenvalues, (Antoulas, 2005). This is especially the case if the system has integrators, which indeed is common in models intended for Kalman filtering. We will therefore extended the proposed solution to handle such systems as well by. This wil be done by first transforming the system matrix into a block triangular form

$$A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \tag{4}$$

where A has been partitioned such that all zero eigenvalues have been placed in A_{22} and all remaining non-zero eigenvalues in A_{11}. Many systems do have such block triangular structure, for example if an observer canonical form has been used. If the system does not have that form, an orthogonal transformation can be applied. The Lyapunov equation corresponding (3b) for this partitioned system will then be

$$\begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}\begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^{\mathsf{T}} & Q_{22} \end{bmatrix} + \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^{\mathsf{T}} & Q_{22} \end{bmatrix}\begin{bmatrix} A_{11}^{\mathsf{T}} & 0 \\ A_{12}^{\mathsf{T}} & A_{22}^{\mathsf{T}} \end{bmatrix} = -\begin{bmatrix} V_{11} & V_{12} \\ V_{12}^{\mathsf{T}} & V_{22} \end{bmatrix},$$

where Q_{T_k} and V_{T_k} have been partitioned in a similar manner as A. This generates as set of in total four Lyapunov and Sylvester equations. The sub-matrices Q_{11} and Q_{12} can now be solved uniquely by solving their corresponding Lyapunv and Sylvester equations, whereas the Lyapunov equation including Q_{22} does not have a unique solution. However, Q_{22} can be solved analytically using the integral (1). By making us of the nilpotent property of A_{22} the matrix exponential in (1) can be expanded with finite number of terms. The resulting expression can be integrated analytically resulting in the following expression

$$Q_{22} = \sum_{i=0}^{p-1}\sum_{j=0}^{p-1} \frac{T_k^{i+j+1}}{i!j!(i+j+1)} A_{22}^{i} S_{22} A_{22}^{j}{}^{\mathsf{T}}. \tag{5}$$

3. NUMERICAL EVALUATION

In this section the numerical properties of the proposed solution will be compared with a standard solution presented by Van Loan (1978). The method is based on a matrix exponential of an augmented $2n \times 2n$ matrix

$$\boldsymbol{e}^{HT_k} = \begin{bmatrix} M_{11} & M_{12} \\ 0 & M_{22} \end{bmatrix}, \qquad H = \begin{bmatrix} A & S \\ 0 & -A^{\mathsf{T}} \end{bmatrix}. \tag{6a}$$

where F_{T_k} and Q_{T_k} are given as

$$F_{T_k} = M_{11}, \qquad Q_{T_k} = M_{12}M_{11}^{\mathsf{T}}. \tag{6b}$$

3.1 Simulation results

In total 100 systems of order $n = 6$ with $m = 4$ stable poles and $p = 2$ additional integrators are randomly generated. Each system is normalized such that the fastest pole is at distance 1 from the imaginary axis, i.e. $\max(|\mathrm{Re}(\lambda_i)|) = 1$. An estimate $\hat{Q}_{T_k}$ is computed using both the proposed method and van Loan's with single precision for different values of the sampling time T_k. Finally, the error

$$\varepsilon = \|\hat{Q}_{T_k} - Q_{T_k}\|_2 / \|Q_{T_k}\|_2$$

is evaluated, where Q_{T_k} is computed using numerical integration of (1) with double precision, here considered as the true value. The result is presented in Figure 1.

According to the result the proposed method outperforms the standard method for large T_k. The reason will become clear if we investigate the two methods further. In van Loan's method (6), both AT_k and $-A^{\mathsf{T}}T_k$ are present in the augmented matrix HT_k and the task to compute its matrix exponential (6a) will become ill-conditioned if T_k or $\max(|\mathrm{Re}(\lambda_i)|)$ is large. In fact, the error will grow exponentially with T_k, or the magnitude of work will grow linearly with T_k to keep a certain tolerance (Van Loan, 1978). This issue is not present in the proposed method.

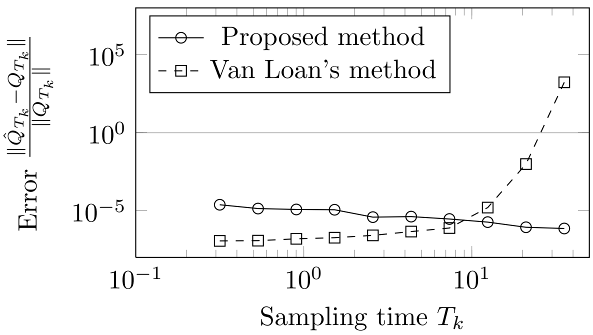

Fig. 1. The performance of the proposed method and Van Loan's method.

However, for short sampling times the proposed method performers slightly worse. This is especially the case if the system has integrators as well as non-zero poles close to the origin leading to that the Sylvester equation corresponding to Q_{12} will become ill-conditioned. Future work shall focus on techniques to circumvent this problem. The proposed method has also advantages when it comes to computational complexity since it only needs to compute the matrix exponential of an $n \times n$ matrix rather than of an augmented $2n \times 2n$ matrix as required by van Loan's method.

4. CONCLUSIONS AND FUTURE WORK

An algorithm for computing an integral involving the matrix exponential common in optimal sampling was proposed. The algorithm is based on a Lyapunov equation and is justified with a novel lemma. An extension to systems with integrators was presented. Numerical evaluations showed that the proposed algorithm has advantageous numerical properties for large sampling times in comparison with a standard method in the literature.

Further work includes extending the algorithm further to handle arbitrary matrices, i.e. also matrices with non-zero eigenvalues mirrored in the imaginary axis. Also the numerical properties should be analyzed further and strategies for improving the numerical properties for slow poles should be investigated.

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