def $\bar{x} = 0$ is

- uniformly stable if $\forall \epsilon > 0 \ \exists \delta(\epsilon) > 0$ independent of t_0 s.t.

  $\|x(t_0)\| < \delta \Rightarrow \|x(t)\| < \epsilon \quad \forall t \geq t_0$

- uniformly asymptotically stable if it is uniformly stable and $\exists \bar{\epsilon} > 0$ independent of t_0 s.t.

  $\|x(t_0)\| < \bar{\epsilon} \rightarrow \lim_{t \to \infty} x(t) = 0$ unif. in t_0

  i.e. for each $\eta > 0 \ \exists T = T(\eta) > 0$ s.t. $\|x(t)\| < \eta \ \forall t > t_0 + T(\eta)$ $\forall \|x(t_0)\| < c$

- globally uniformly ~~stab~~ asymptotically stable if it is uniformly stable and $\forall x(t_0)$

  $\lim_{t \to \infty} x(t) = \bar{x} = 0$ uniformly in t_0

  i.e. for each pair $\eta, c > 0 \ \exists T = T(\eta, c) > 0$ s.t.

  $\|x(t)\| < \eta \quad \forall t \geq t_0 + T(\eta, c) \quad \forall \|x(t_0)\| < c$

example $\dot{x} = -\frac{x}{1+t}$

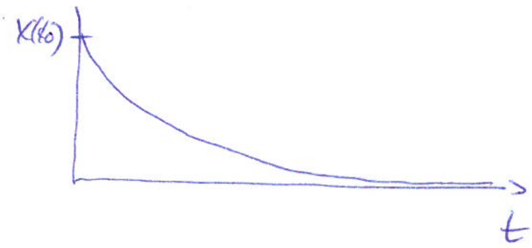
solution $x(t) = x(t_0)\exp\left(\int_{t_0}^{t} \frac{-1}{1+s}\, ds\right) = x(t_0)\frac{1+t_0}{1+t}$

$|x(t)| < |x(t_0)| \quad \forall t \geq t_0 \Rightarrow \bar{x}=0$ is stable.

clearly $x(t) \to 0$ as $t \to \infty$

but convergence to 0 is not uniform w.r.t. t_0

if $t_0 = 0$

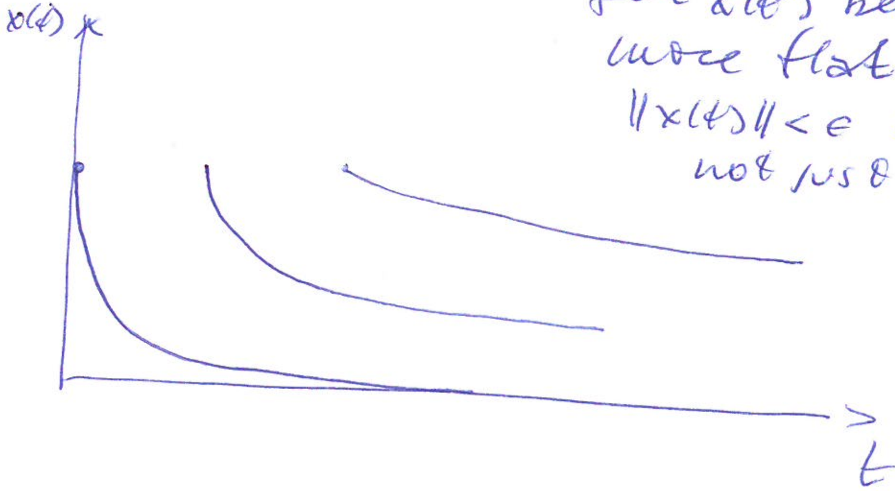


$x(t) = x(t_0)\frac{1}{1+t}$

~~if $t_0 = 10$~~

~~$x(t) = x(t_0)\frac{(1+t_0)}{1+t}$~~

changing t_0



as t_0 grows the curve for $x(t)$ becomes more and more flat $\Rightarrow$ for a given ϵ δ such that $\|x(t)\| < \epsilon$ depends also on t_0 not just on ϵ

to better understand these definitions: comparison functions

def.
- **class-K function**

  $\alpha : [0\ a) \to [0\ \infty)$, C^0, is of class K if $\alpha(0) = 0$ and α strictly increasing

- **class K_∞ function**

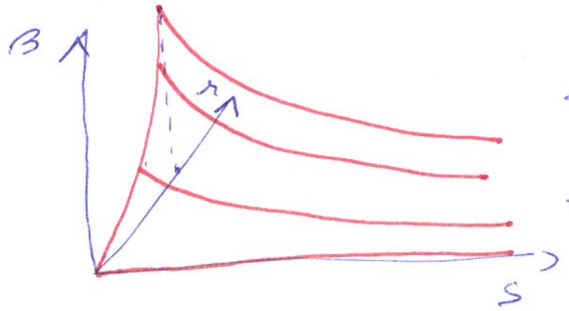
  A class K function is said to be K_∞ if $a = +\infty$ and $\lim_{r \to \infty} \alpha(r) = \infty$

- **class $K\mathcal{L}$ function**

  $\beta : [0\ a) \times [0, \infty) \to [0, \infty)$, C^0, is of class $K\mathcal{L}$ if
  - for all fixed s, $\beta(r,s)$ is of class K w.r.t. r
  - for all fixed r, $\beta(r,s)$ is decreasing w.r.t. s and $\lim_{s \to \infty} \beta(r,s) = 0$

examples
- $\alpha(r) = r^c$ $(c > 0)$ is K_∞ (strictly increasing in $[0\ a)$)
- $\alpha(r) = \mathrm{atan}(r)$ is K but not K_∞
- all class K functions are invertible in $[0\ a)$
- $\beta(r,s) = r^c e^{-t}$ is $K\mathcal{L}$

meaning of $K\mathcal{L}$



- increasing in r (norm of x)
- decreasing in s (time)

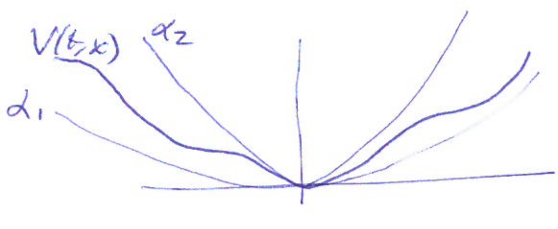
Prop. of comparison funct.

- α is K (K_∞) $\Rightarrow$ α^{-1} is K (K_∞)
- α is K $\Rightarrow$ it is invertible in $[0, a)$
- α_1, α_2 are K $\Rightarrow$ $\alpha_1 \circ \alpha_2$ is K
- $V: \mathcal{D} \to \mathbb{R}$ pos. def. $\Rightarrow$ $\exists\ \alpha_1, \alpha_2$ of class K s.t. $\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|)$ in B_r
- $V: \mathbb{R}^n \to \mathbb{R}$ pos. def, radially unbounded $\Rightarrow$ $\exists\ \alpha_1, \alpha_2$ of class K_∞ s.t.

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|) \quad \text{in } \mathbb{R}^n$$

example: $V(x) = x^T P x \quad P = P^T > 0$

$$\Rightarrow \alpha_1(\|x\|) = \lambda_{min}(P)\|x\|_2^2 \leq V(x) \leq \lambda_{max}(P)\|x\|_2^2 = \alpha_2(\|x\|)$$



class K funct. are a way to guarantee pos. def. for time-varying $V(t,x)$

- For $y \in \mathbb{R}_+$, α of class K, the scalar system

$$\dot{y} = -\alpha(y) \quad \text{Lipsch. cont. on } [0, a)$$
$$y(t_0) = y_0 \qquad y_0 \in [0, a)$$

has solution $y(t) = \beta(y_0, t - t_0) \quad \forall t$

where β is of class $K\mathcal{L}$ $\Rightarrow$ converges to 0 as $t - t_0 \to \infty$

example: $\dot{y} = -ky \Rightarrow y(t) = e^{-k(t-t_0)} y_0 \Rightarrow \beta(r, s) = e^{-ks} r$

Comparison functions are used to give more transparent conditions (characteriz.) of uniform stab and unif. asympt. stab.

<u>Lemma</u> $\bar{x}=0$ is

- uniformly stable iff $\exists$ a class K function α and const $c>0$ (indep. of t_0) s.t. $\forall \|x(t_0)\|<c$ it is $\|x(t)\| \leq \alpha(\|x(t_0)\|)$ $\forall t \geq t_0$

- uniformly asympt. stable iff $\exists$ a class KL function β and a const $c>0$ (indep. of t_0) s.t. $\forall$ $\|x(t_0)\|<c$ it is
$$\|x(t)\| \leq \beta(\|x(t_0)\|, t-t_0) \quad \forall t \geq t_0$$

When the KL function above has the form
$$\beta(r,s) = k r e^{-\lambda s}$$
then we have a "stronger type" of unif. as. stab.:

<u>def</u> $\bar{x}=0$ is <u>exponentially stable</u> if $\exists$ constants $c, k, \lambda > 0$ s.t.
$$\|x(t)\| \leq k\|x(t_0)\| e^{-\lambda(t-t_0)} \quad \forall \|x(t_0)\|<c$$
and <u>globally exponentially stable</u> if this holds $\forall c>0$

- expon. stab $\Rightarrow$ linear-like convergence rate

Example of non-autonomous system: <u>Switching syst.</u>

A switching syst. is a collection of systems in $\mathbb{R}^n$ $f_1(x) \dots f_m(x)$
indexed by an indicator function

$$\dot{x} = f_p(x) \quad p \in \mathcal{P} = \text{index set} = \{1, \dots, m\}$$

when we choose a function for p (piecewise const. and containing a finite number of discontinuities)

$$\sigma : \underset{\text{time}}{[0, \infty)} \longrightarrow \underset{\text{index set}}{\mathcal{P}}$$

ex:

$$\sigma = \begin{cases} 3 & t \in [0, 2) \\ 1 & t \in [2, 5) \\ 4 & t \in [5, 7) \\ 1 & \dots \\ \vdots & \end{cases}$$

indicator of which of the vector fields in the family is taken at each time instant

switching times = jumps in σ

$\Rightarrow \dot{x}(t) = f_{\sigma(t)}(x(t))$ is a time varying system.
$\Rightarrow$ non-autonomous

- If switching times "do not accumulate" in time and each $f_p(x)$ is Lipschitz cont., then also
$\dot{x}(t) = f_{\sigma(t)}(x(t))$ is Lipschitz cont
$\Rightarrow$ solution exists unique.

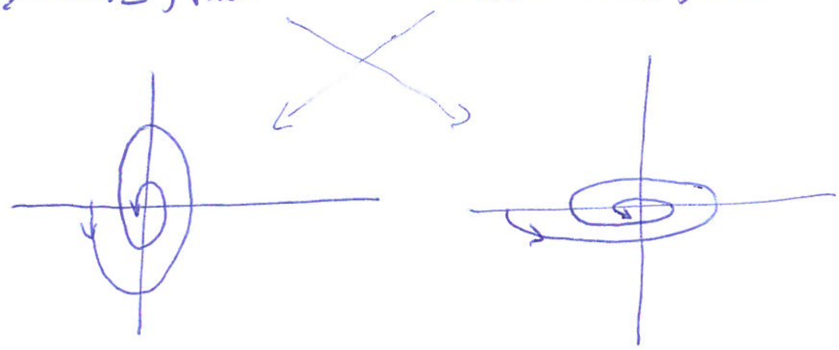
· because of the time variance, even if each vector field $f_1(x) \dots f_m(x)$ is asympt. st. the switching system need not be asympt. st.

<u>example</u>

switching linear system, $m=2$

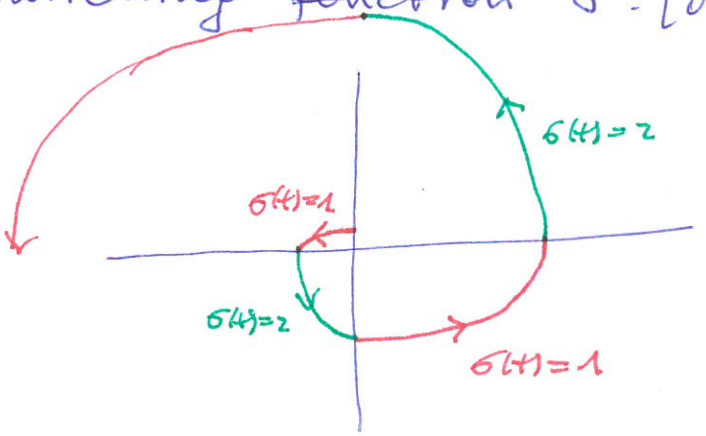
$$A_1 = \begin{bmatrix} -1 & 10 \\ -100 & -1 \end{bmatrix} \qquad A_2 = \begin{bmatrix} -1 & 100 \\ -10 & -1 \end{bmatrix}$$

$$\lambda_{12} = -1 \pm j\sqrt{10^3} \qquad \lambda_{12} = -1 \pm j\sqrt{10^3}$$



$\dot{x} = A_2 x$ $\qquad\qquad$ $\dot{x} = A_1 x$

$\Rightarrow \exists$ a switching ~~function~~ signal $\sigma : [0, \infty) \to \{1, 2\}$ s.t.



$\Rightarrow$ system diverges

$\Rightarrow$ stability of each mode is <u>not enough</u> to guarantee stability of the switched system.

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concept that one needs here is uniform asympt. stab.

def the switched system $\dot{x} = f_p(x)$ $p \in \mathcal{P}$ is unif. asympt. stable if $\exists\ c > 0$ and $\beta \in \mathcal{KL}$ s.t. $\forall$ switching signals $\sigma : [0, \infty) \to \mathcal{P}$ the solutions of $\dot{x} = f_{\sigma(t)}(x(t))$ are s.t.

$$\|x(0)\| < c \Rightarrow \|x(t)\| \leq \beta(\|x(t)\|, t) \quad \forall t \geq 0$$

def uniform asympt. stab. is global if it holds $\forall\ x(0)$.

Generalizing Lyapunov direct method: time-varying Lyapunov functions

Thm Let $\bar{x}=0$ be equil point of $\dot{x}=f(t,x)$ $0\in\mathcal{D}$. Let $V:[t_0,\infty)\times\mathcal{D}\to\mathbb{R}$ be C^1 funct. such that

$$W_1(x)\leq V(t,x)\leq W_2(x)$$

$$\dot{V}(t,x)=\frac{\partial V}{\partial t}+\frac{\partial V}{\partial x}f(t,x)\leq 0 \qquad (*)$$

$\forall t\geq t_0$, $\forall x\in\mathcal{D}$, where $W_1(x)$, $W_2(x)$ C^0 pos. def in $\mathcal{D}$. Then $\bar{x}=0$ is uniformly stable

If instead of $(*)$ we have

$$\frac{\partial V}{\partial t}+\frac{\partial V}{\partial x}f(t,x)\leq -W_3(x)$$

where $W_3(x)$ C^0 pos. def. in $\mathcal{D}$, then $\bar{x}=0$ is uniformly asympt. stable.

Finally, if $\mathcal{D}=\mathbb{R}^n$ and $W_1(x)$ is radially unbounded, then $\bar{x}=0$ is globally uniformly asympt. stable.

Proof

Construct a bounded invariant set in which V is decreasing. From B_r, consider

a β s.t. $\beta < \min_{\|x\|=r} W_1(x)$

and define sets

$\Omega_V = \{ x \in B_r \text{ s.t. } V(t,x) \leq \beta \}$ (a time varying set)

$\Omega_i = \{ x \in B_r \text{ s.t. } W_i(x) \leq \beta \}$ $i = 1, 2$

Ω_V is sandwitched between sets that do not depend on time → can treat it!

$\Rightarrow \Omega_2 \subset \Omega_V \subset \Omega_1 \subset B_r$

Assume $x(t_0) \in \Omega_2 \Rightarrow x(t_0) \in \Omega_V$

$\Rightarrow V(t_0, x(t_0)) \leq \beta \Rightarrow V(t, x(t)) \leq \beta \quad \forall t$

Choose class K functions α_1, α_2 s.t.

$\alpha_1(\|x\|) \leq W_1(x) \leq V(t,x) \leq W_2(x) \leq \alpha_2(\|x\|) \quad \forall t$

Since $\dot{V}(t,x) \leq 0$, $\Rightarrow V(t, x(t)) \leq V(t_0, x(t_0))$

$\Rightarrow \alpha_1(\|x\|) \leq V(t, x(t)) \leq V(t_0, x(t_0)) \leq \alpha_2(\|x(t_0)\|)$

apply α_1^{-1} to all inequal

$\|x\| \leq \underbrace{\alpha_1^{-1}(\alpha_2(\|x(t_0)\|))}_{\text{class K function}}$

from previous Lemma, uniform stability follo

To show uniform asympt. stab:

take α_3 of class K s.t. $\alpha_3(\|x\|) \leq W_3(x)$

then

$$\dot{V}(t,x) = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \leq -W_3(x) \leq -\alpha_3(\|x\|)$$

from before:

$$V \leq \alpha_2(\|x\|) \Leftrightarrow \alpha_2^{-1}(V) \leq \|x\|$$

$$\Leftrightarrow \alpha_3(\alpha_2^{-1}(V)) \leq \alpha_3(\|x\|)$$

$$\Rightarrow \dot{V} \leq -\alpha_3(\alpha_2^{-1}(V))$$

$\Rightarrow$ V is bounded above by the scalar system (comparison Lemma)

$$\dot{z} = \underbrace{-\alpha_3(\alpha_2^{-1}}_{\text{class } K \text{ function}}(z))$$

for this system, the solution is a class $K\mathcal{L}$ function σ, s.t. $z \to 0$ as $t \to \infty$

$$V(t, x(t)) \leq \underbrace{\sigma}_{K\mathcal{L} \text{ funct.}}(V(t_0, x(t_0)), t - t_0)$$

$\Rightarrow$ V must decrease to 0, and from above

$$\|x(t)\| \leq \alpha_1^{-1}(V(t, x(t))) \leq \alpha_1^{-1}(\sigma(V(t_0, x(t_0)), t - t_0))$$

$$\leq \underbrace{\alpha_1^{-1}(\sigma(\alpha_2(\|x(t_0)\|, t - t_0)))}_{\text{class } K\mathcal{L} \text{ function}} \Rightarrow \|x(t)\| \to 0 \text{ as } t \to \infty$$

When the functions $W_i(x)$ are norms of $\|x\|$,
i.e. $W_i(x) = \|x\|^a$ for some $a > 0$ constant
then we have (local) exponential stability

Thm

Let $\bar{x} = 0$ be equil p. of $\dot{x} = f(t,x)$, $0 \in \mathcal{D}$
Let $V : [0,\infty) \times \mathcal{D} \to \mathbb{R}$ be C^1 function s.t.

$$k_1 \|x\|^a \le V(t,x) \le k_2 \|x\|^a$$

$$\dot{V}(t,x) = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \le -k_3 \|x\|^a$$

for some k_1, k_2, k_3, a posit. const.
then $\bar{x} = 0$ is locally exp. stable (or global if all assumptions hold globally).

Proof From $\frac{V(t,x)}{k_2} \le \|x\|^a$ one gets $\dot{V}(t,x) \le -\frac{k_3}{k_2} V(t,x)$

$\Rightarrow$ from comparison Lemma, V bounded above by

$$V(t, x(t)) \le V(t_0, x(t_0))\, e^{-\frac{k_3}{k_2}(t-t_0)}$$

$$\Rightarrow \|x(t)\| \le \left(\frac{k_2}{k_1}\right)^{1/a} \|x(t_0)\|\, e^{-\frac{k_3}{a k_2}(t-t_0)}$$

$$\|x(t)\| \le \left(\frac{V(t,x)}{k_1}\right)^{1/a} \le \left(\frac{V(t_0, x(t_0))\, e^{-\frac{k_3}{k_2}(t-t_0)}}{k_1}\right)^{1/a} \le \left(\frac{k_2 \|x(t_0)\|^a\, e^{-\frac{k_3}{k_2}(t-t_0)}}{k_1}\right)^{1/a}$$

example $\begin{cases} \dot{x}_1 = -x_1 - g(t)\, x_2 \\ \dot{x}_2 = x_1 - x_2 \end{cases}$ $\bar{x} = 0$ equil

where $g(t)$ C^1 s.t. $0 \le g(t) \le k$, $\dot{g}(t) \le g(t)$ $\forall t$

Take $V(t,x) = x_1^2 + (1+g(t))\,x_2^2$ as candid. Lyap. funct.

$\Rightarrow x_1^2 + x_2^2 = \|x\|_2^2 \le V(t,x) \le x_1^2 + (1+k)\,x_2^2 \quad \forall x \in \mathbb{R}^2$
$\le (1+k)\|x\|_2^2$

$\Rightarrow V(t,x)$ pos. def., decrescent in time, radially unbound.
bounded by quadratic functions

$$\dot{V}(t,x) = 2x_1\dot{x}_1 + \dot{g}(t)x_2^2 + 2(1+g(t))x_2\dot{x}_2$$
$$= 2x_1(-x_1 - g(t)x_2) + \dot{g}(t)x_2^2 + 2(1+g(t))\,x_2(x_1 - x_2)$$
$$= -2x_1^2 - \cancel{2g(t)x_1x_2} + \dot{g}(t)x_2^2 + 2x_1x_2 + \cancel{2g(t)x_1x_2} - 2x_2^2 - 2g(t)x_2^2$$
$$= -2x_1^2 - (2 + 2g(t) - \dot{g}(t))\,x_2^2 + 2x_1x_2$$

Since $2 + 2g(t) - \dot{g}(t) \ge 2 + 2g(t) - g(t) \ge 2$

$$\dot{V}(t,x) \le -2x_1^2 - 2x_2^2 + 2x_1x_2 = -[x_1\ x_2]\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -x^T Q x$$

with $Q = Q^T > 0$ pos. def

$\Rightarrow \dot{V}(t,x)$ neg. def. $\Rightarrow \bar{x} = 0$ (globally) unif. asympt. st.

Since $V(t,x)$ bounded by quadratic funct. and also $\dot{V}(t,x)$ then $\bar{x} = 0$ is (globally) exponentially asympt. stable.

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For a linear system: asymptotic stability is always exponential

$\dot{x} = Ax$ $\quad x(t) = e^{At}x_0 \Rightarrow \|x(t)\| \leq k e^{\max_i \{Re[\lambda_i]\} t} \|x_0\|$

where $\max Re[\lambda_i] < 0$ because of asympt. st.

- For an autonomous nonlinear system, if Jacobian linearization is asympt. stable then $\bar{x} = 0$ of the nonlinear system is locally exponentially stable (we said earlier asympt. stable) (It is iff cond, see below)

  to show it: use quadratic Lyapunov eq., which is bounded by min/max eig of P $\quad V = x^T P x$

  with $\lambda_{min}(P)\|x\|^2 \leq V(x) \leq \lambda_{max}(P)\|x\|^2$

- For linear time-varying systems

$$\dot{x}(t) = A(t)x(t)$$

uniform asymptotic stability $\Leftrightarrow$ exponential stab.

provided that $A(t)$ is bounded and Lipschitz

However it cannot be characterized by the eigenvalues of $A(t)$! (It could be $Re[\lambda_i(t)] <$ and $\bar{x} = 0$ unstable

One can use instead the time-varying Lyapunov function $V(t,x) = x^T P(t) x$

where $0 < c_1 I \leq P(t) \leq c_2 I \quad \forall t$

and $\dot{P}(t) = A^T(t)P(t) + P(t)A(t) + Q(t)$

with $Q(t) = Q^T(t) \geq c_3 I > 0 \quad \forall t$

- Then nonlinear non-autonomous system

$$\dot{x} = f(t,x)$$

has $\bar{x}=0$ locally exponentially stable <u>iff</u>

$$\dot{x} = A(t)x \quad \text{with } A(t) = \frac{\partial f(t,x)}{\partial x}\Big|_{\bar{x}=0}$$

bounded and Lipschitz, uniform in t

has $\bar{x}=0$ as an exponentially stable equil (i.e. uniformly asympt. st.)

<u>Summary</u>:

AUTONOMOUS	NON-AUTONOMOUS
$\dot{x} = f(x)$	$\dot{x} = f(t,x)$
$\bar{x}=0$ loc. asympt. st	ok ~~$\bar{x}=0$ is loc. unif. as. st~~ ok!
$\Uparrow$	~~$\Longleftrightarrow$~~ $\Uparrow$
$\bar{x}=0$ loc exp. st.	$\bar{x}=0$ is loc. exp. stable
$\Longleftrightarrow$ see p. 166	see p. 163 $\Longleftrightarrow$
$A = \frac{\partial f}{\partial x}\Big\vert_{\bar{x}=0}$ is as. st.	$A(t) = \frac{\partial f(t,x)}{\partial x}\Big\vert_{\bar{x}=0}$ is loc. unif. as. st.
$\Longleftrightarrow$	$\Longleftrightarrow$
$A = \frac{\partial f}{\partial x}\Big\vert_{\bar{x}=0}$ is exp. st.	$A(t) = \frac{\partial f(t,x)}{\partial x}\Big\vert_{\bar{x}=0}$ is loc. exp. st.

Autonomous system

nonlinear

$\dot{x} = f(x)$

Jacobian linearization

$\dot{x} = Ax \quad A = \left.\frac{\partial f}{\partial x}\right|_0$

$\bar{x}=0$ loc. as. st. $\Longleftarrow$ (from Lyap. ind. meth) $\bar{x}=0$ as. st.

$\Uparrow$ $\Updownarrow$

$\bar{x}=0$ loc. exp. st. $\Longleftrightarrow$ $\bar{x}=0$ as. st.

$\bar{x}=0$ loc. exp. st. $\Longleftrightarrow$ $\bar{x}=0$ exp. st.

Non autonomous system

nonlinear

$\dot{x} = f(t,x)$

Jacobian linearization

$\dot{x} = A(t)x \quad A(t) = \left.\frac{\partial f(t,x)}{\partial x}\right|_0$

$\bar{x}=0$ loc. unif. as. st. $\Longleftarrow$ $\bar{x}=0$ unif. as. st.

$\Uparrow$ $\Updownarrow$

$\bar{x}=0$ loc. exp. st. $\Longleftrightarrow$ $\bar{x}=0$ unif. as. st.

$\bar{x}=0$ loc. exp. st. $\Longleftrightarrow$ $\bar{x}=0$ exp. st.

Stability of perturbed systems (Khalil 9.1)

Consider a system with <u>vanishing perturbation</u>

$$\dot{x} = f(t,x) + g(t,x)$$

where vanishing means $g(t,0) = 0$

If the nominal system $\dot{x} = f(t,x)$ has an exponentially stable equil point, then it is robust to ~~pertorbations~~ perturbations which are bounded

<u>thm</u> Let $\bar{x}=0$ be an exponentially stable equil of $\dot{x} = f(t,x)$. Let $V(t,x)$ be Lyapunov funct. $V: [0,\infty) \times \mathcal{D} \to \mathbb{R}$, C^1, s.t.

$$c_1 \|x\|^2 \le V(t,x) \le c_2 \|x\|^2$$

$$\dot{V}(t,x) = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \le -c_3 \|x\|^2$$

$\forall x \in \mathcal{D}$
$\forall t \in [0,\infty)$
for some $c_i > 0$

$$\left\| \frac{\partial V}{\partial x} \right\| \le c_4 \|x\|$$ ← extra growth bound

Suppose the perturbation term $g(t,x)$ satisfies

$$\|g(t,x)\| \le \gamma \|x\| \quad \forall t \ge 0, \forall x \in \mathcal{D}$$

with $\gamma < c_3/c_4$

then $\bar{x}=0$ is an exponentially stable equil of $\dot{x} = f(t,x) + g(t,x)$

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Proof compute the total derivative of V:

$$\dot{V}(t,x) = \underbrace{\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x)}_{\substack{\dot{V} \text{ for nominal} \\ \text{system } \leq -c_3 \|x\|^2}} + \underbrace{\frac{\partial V}{\partial x} g(t,x)}_{\text{effect of the perturbation}}$$

with the growth bound $\left\| \frac{\partial V}{\partial x} \right\| \leq c_4 \|x\|$ we can make a worst-case analysis bounding $\frac{\partial V}{\partial x} g$ by a nonneg. term.

$$\Rightarrow \dot{V}(t,x) \leq -c_3 \|x\|^2 + \left\| \frac{\partial V}{\partial x} \right\| \|g(t,x)\| \leq -c_3 \|x\|^2 + c_4 \gamma \|x\|^2$$

If $\gamma < \frac{c_3}{c_4}$ $\Rightarrow \dot{V}(t,x) \leq \underbrace{(-c_3 + c_4 \gamma)}_{<0} \|x\|^2 \leq \xi \|x\|^2$

//

example $\dot{x} = Ax + g(t,x)$

A Hurwitz / g s.t. $\|g(t,x)\|_2 \leq \gamma \|x\|_2$ $\forall t \geq 0$ $\forall x \in \mathbb{R}^2$

take $Q = Q^T > 0$ and solve

$$PA + A^T P = -Q \quad \Rightarrow P = P^T > 0$$

use $V(x) = x^T P x$ as Lyapunov function for it it is $\lambda_{min}(P)\|x\|_2^2 \le V(x) \le \lambda_{max}(P)\|x\|_2^2$

for the nominal system:

$$\frac{\partial V}{\partial x} Ax = -x^T Q x \le -\lambda_{min}(Q)\|x\|_2^2$$

plus

$$\left\|\frac{\partial V}{\partial x}\right\|_2 = \|2x^T P\|_2 \le 2\|P\|_2 \|x\|_2 = 2\lambda_{max}(P)\|x\|_2$$

hence $\dot{V}$ for the perturbed system

$$\dot{V}(x) \le -\lambda_{min}(Q)\|x\|_2^2 + 2\gamma\lambda_{max}(P)\|x\|_2^2$$

$$= \left(-\lambda_{min}(Q) + 2\gamma\lambda_{max}(P)\right)\|x\|_2^2$$

if $\gamma < \dfrac{\lambda_{min}(Q)}{2\lambda_{max}(P)}$ it is $\dot{V} < 0$

$\Rightarrow$ $\bar{x}=0$ is exponentially stable for the perturbed system.

If $Q = I$, the ratio $\dfrac{\lambda_{min}(Q)}{\lambda_{max}(P)}$ is maximized

(i.e. I can take the largest perturbation)

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- when $\bar{x}=0$ is uniformly asympt. stable (but not exponentially for the nominal system) then stability of perturbed system is more involved
- when perturbation is not vanishing but just bounded, then stability is more involved also...

Converse theorems

Converse theorems should answer the following:

1) if equil is stable (asympt. st, unif. as. st, exp. s) does it always $\exists$ a Lyapunov funct. V?

2) How to construct it?

Only the first question can be answered satisfactorily, but not the construction (except for e.g. linear systems or other special classes)

A typical converse thm (similar results hold for uniform asympt. stability)

<u>thm</u> Let $\bar{x}=0$ be equil point for $\dot{x}=f(t,x)$ with $f:[0,\infty)\times\mathcal{D}\to\mathbb{R}^n$, C^1.

Assume

1) $\frac{\partial f}{\partial x}$ bounded in B_r (for some r), uniformly bounded in t

2) $\bar{x}$ is exponentially stable: $\exists\, k, \lambda, r_0 > 0$ s.t.

$\|x(t)\| \leq k\|x(t_0)\| e^{-\lambda(t-t_0)}$ $\forall\, x(t_0)\in B_{r_0}$ $\forall\, t\geq t_0\geq 0$

there $\exists$ a C^1 function $V:[0,\infty)\times B_0 \to \mathbb{R}$ s.t.

$$c_1\|x\|^2 \leq V(t,x) \leq c_2\|x\|^2$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \leq -c_3\|x\|^2$$

$$\left\|\frac{\partial V}{\partial x}\right\| \leq c_4\|x\| \quad \text{for some } c_1, c_2, c_3, c_4 > 0$$

Switching systems

how do we use uniform asympt. stab. for the switching system

$$\dot{x} = f_p(x) \qquad p \in \mathcal{P}$$

{time dependence is in the switching signal $\sigma : [0,\infty) \to \mathcal{P}$}
~~Alpos~~ not explicitly in the $f_1(x) \dots f_m(x)$

A possibility is to look for a common Lyapunov function i.e. a Lyapunov function valid for all the m vector fields $f_1(x), \dots, f_m(x)$.

Since each $f_p(x)$ is time-indep. then also $V(x)$ can be chosen time-indep. However as in a non-autonomous system $\frac{\partial V}{\partial x} f_p(x)$ has to be bounded above by $-W(x)$ with $W(x)$ p.d. Saying $\frac{\partial V}{\partial x} f_p(x) < 0$ is not enough.

def $V : \mathbb{R}^n \to \mathbb{R}$ C^1, pos. def is a common Lyapunov function for the switched system $\dot{x} = f_p(x)$ $p \in \mathcal{P}$ if $\exists$ $W : \mathbb{R}^n \to \mathbb{R}$, C^0, W pos. def. s.t. $\frac{\partial V}{\partial x} f_p(x) \leq W(x)$ $\forall x$ $\forall p \in \mathcal{P}$

if $\mathcal{P}$ = finite set or a compact $\frac{\partial V}{\partial x} f_p(x) \leq 0$ $\forall x$ $\forall p \in \mathcal{P}$ is enough!

suff. cond. for global uniform asympt. stab.

Thm If in the switching system $\dot{x} = f_p(x)$ $p \in \mathcal{P}$ all $f_p(x)$ share a radially unbounded common Lyapunov function then the switching system is globally uniformly asympt. st.

Proof: very similar to Lyapunov direct method (Having $\frac{\partial V}{\partial x} f_p(x) \leq -W(x)$ with $W(x)$ guarantees that the rate of decrease of $V(x)$ never vanishes even if switching happens in $\mathcal{P}$ non-compact)

How to construct a common Lyapunov funct?
In general it is difficult.
For switching linear system it corresponds to solving m Lyapunov eq. simultaneously by the same P

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ex : switched linear system

$$f_p(x) = A_p x \qquad p \in \mathcal{P} = \{1, \dots, m\}$$

search for $V(x) = x^T P x$ (common quadratic Lyapunov function) → CQLF
$P = P^T > 0$

s.t. $\begin{cases} A_k^T P + P A_k \leq -Q \\ P = P^T > 0 \end{cases}$, $Q = Q^T > 0 \quad \forall k \in \mathcal{P}$

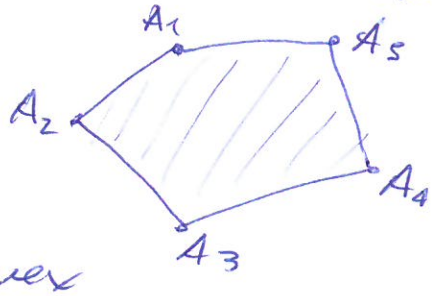
on a compact

$\begin{cases} A_k^T P + P A_k \leq 0 \\ P = P^T > 0 \end{cases}$

is enough.

P can be found by LMI

intuitive meaning of CQLF : if the same QLF holds for all vertices $A_1, \dots A_m$ of the polytope,



then it has to hold for all convex combinations inside the polytope. $\dot{x} = A_\sigma(x)$ can be anything inside the polytope.

All convex comb. $\dot{x} = (1-\alpha) A_1 x + \alpha A_2 x$ must be asympt. stable.

$$\dot{x} = \sum_{i=1}^{m} \alpha_i A_i x \qquad \begin{array}{l} \alpha_i \geq 0 \\ \alpha_i \leq 1 \\ \sum \alpha_i = 1 \end{array}$$