meaning: if I have the system

$$\dot{x} = A_1 x \Rightarrow x(t) = e^{A_1 t} x_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos t & -\sin t \\ 0 & \sin t & \cos t \end{bmatrix} x_0$$

$\Rightarrow$ evolutions are rotations around the x_1 axis

If instead $\dot{x} = A_2 x \Rightarrow x(t) = \begin{bmatrix} \cos t & 0 & \sin t \\ 0 & 1 & 0 \\ -\sin t & 0 & \cos t \end{bmatrix} x_0$

$\Rightarrow$ rotations around the x_2 axis

If instead $\dot{x} = A_3 x \Rightarrow x(t) = \begin{bmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{bmatrix} x_0$

$\Rightarrow$ rotations around the x_3 axis

$\Rightarrow so(3)$ contains all infinitesimal rotations in $\mathbb{R}^3$

xample set of all n×n matrices is a Lie algebra called general linear algebra $gl(n) = \{A \in \mathbb{R}^{n\times n}\}$ with matrix commutator as ~~Lie bracket~~ bilinear operation op.

$[A, B] \in gl(n) \Rightarrow [A, B] \in gl(n)$

example $\mathcal{V}(M)$ = set of all v.f. over M is a Lie algebra with Lie bracket as bilinear operation
this is an ∞-dim. Lie algebra.

Meaning of Lie bracket

Assume we have two complete v.f $f, g \in \mathcal{V}(M)$ and consider the associate flow

$$\Phi^f_t : \mathbb{R} \times M \longrightarrow M$$
$$(t, p) \mapsto \Phi^f_t(p)$$

we know that f is the infinitesimal generator of the flow

$$f(p) = \frac{d}{dt} \Phi^f_t(p) \Big|_{t=0}$$

thm $$[f,g] = \frac{\partial^2}{\partial s \partial t}\left(\Phi^{-g}_s \circ \Phi^{-f}_t \circ \Phi^{g}_s \circ \Phi^{f}_t\right)\Big|_{t=s=0}$$

proof Consider an integral curve followed for a time $t=\epsilon$ (small).

From $$\frac{d}{dt}\Phi^f_t(x_0)\Big|_{t=0} = f(x_0)$$

$$\frac{d^2}{dt^2}\Phi^f_t(x_0)\Big|_{t=0} = \frac{d}{dt}f(x_0) = \frac{\partial f(x_0)}{\partial x}f(x_0)$$

we have, up to second order terms, the Taylor expansion

$$\Phi^f_\epsilon(x_0) \simeq x_0 + \epsilon f(x_0) + \frac{\epsilon^2}{2}\frac{\partial f(x_0)}{\partial x}f(x_0) + O(\epsilon^3)$$

compose this with flow along g for another time ϵ

$$\Phi^g_\epsilon \circ \Phi^f_\epsilon(x_0) \simeq \Phi^g_\epsilon\left(x_0 + \epsilon f(x_0) + \frac{\epsilon^2}{2}\frac{\partial f(x_0)}{\partial x}f(x_0) + O(\epsilon^3)\right)$$

$$= x_0 + \epsilon f(x_0) + \frac{\epsilon^2}{2}\frac{\partial f(x_0)}{\partial x}f(x_0) + O(\epsilon^3) + \epsilon g\left(x_0 + \epsilon f(x_0) + O(\epsilon^2)\right)$$

$$+ \frac{\epsilon^2}{2}\frac{\partial g(x_0 + O(\epsilon))}{\partial x}g(x_0 + O(\epsilon))$$

expanding $g(x_0 + \epsilon f(x_0)) = g(x_0) + \epsilon \frac{\partial g(x_0)}{\partial x} f(x_0) + O(\epsilon^2)$

$$\Phi_\epsilon^g \circ \Phi_\epsilon^f (x_0) = x_0 + \epsilon\left(f(x_0) + g(x_0)\right) + \epsilon^2\left(\frac{1}{2}\frac{\partial f(x_0)}{\partial x} f(x_0) + \right.$$

$$\left. + \frac{\partial g(x_0)}{\partial x} f(x_0) + \frac{1}{2}\frac{\partial g(x_0)}{\partial x} g(x_0)\right) + O(\epsilon^3)$$

now flow along $-f$ for a time ϵ (do the calcul!)

$$\Phi_\epsilon^{-f} \circ \Phi_\epsilon^g \circ \Phi_\epsilon^f (x_0) = x_0 + g(x_0) + \epsilon^2\left(\frac{\partial g(x_0)}{\partial x} f(x_0)\right.$$

$$\left. + \frac{1}{2}\frac{\partial g(x_0)}{\partial x} g(x_0) - \frac{\partial f(x_0)}{\partial x} g(x_0)\right) + O(\epsilon^3)$$

final leg: flow along $-g$

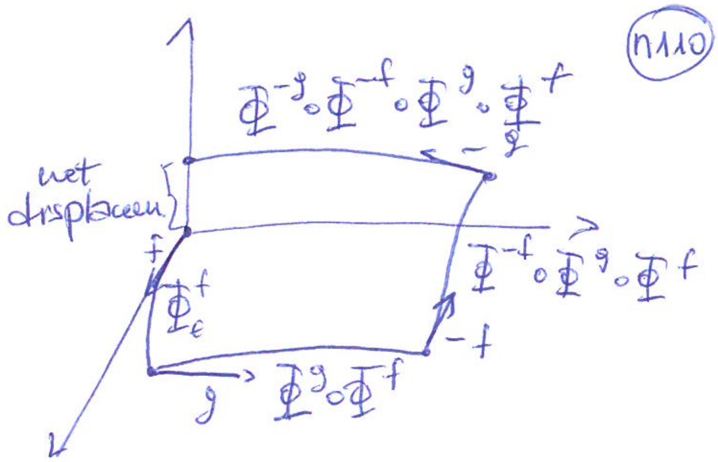
$$\Phi_\epsilon^{-g} \circ \Phi_\epsilon^{-f} \circ \Phi_\epsilon^g \circ \Phi_\epsilon^f (x_0) = \underbrace{x_0}_{\text{initial point}} + \epsilon^2\underbrace{\left(\frac{\partial g(x_0)}{\partial x} f(x_0) - \frac{\partial f(x_0)}{\partial x} g(x_0)\right)}_{\text{displacement = Lie bracket}} + O(\epsilon^3)$$

$$= x_0 + \epsilon^2 [f, g](x_0) + O(\epsilon^3)$$

//

⇒ displacement given by the Lie bracket is a second order term ⇒ it does not appear from linearization arguments!

practical meaning



$f, g \in \mathcal{V}(M)$

$\Rightarrow [f, g] \in \mathcal{V}(M)$

$\Rightarrow$ the "net motion" is still in M

$\Rightarrow$ the Lie bracket determines new directions of motion not present in the original v.f.

$\rightarrow$ effect of noncommutativity of the v.f.

Coroll $f, g \in \mathcal{V}(M)$ commute (i.e. $[f, g] = 0$ $\forall x \in M$)

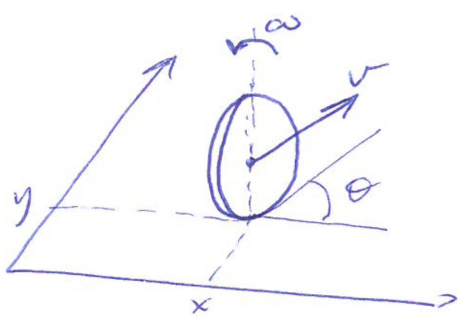
if and only if $\Phi^g_s \circ \Phi^f_t = \Phi^f_t \circ \Phi^g_s$ $\forall t, s \geq 0$ $\forall p \in M$

Prop (commutator of 2 Lie derivatives)

$$[L_f, L_g] = L_f L_g - L_g L_f = L_{[f, g]}$$

proof: straightforward from def of $L_f(L_g \varphi)$

<u>example</u>: rolling wheel



ODE: $\dot{x} = v\cos\theta$

$\dot{y} = v\sin\theta$

$\dot{\theta} = \omega$

v = linear velocity
ω = angular vel.

manifold $M = \mathbb{R}^2 \times S^1$

local coord $\begin{bmatrix} x \\ y \\ \theta \end{bmatrix} \triangleq z \Rightarrow$ local coord. on T_pM $\left(\frac{\partial}{\partial x} \; \frac{\partial}{\partial y} \; \frac{\partial}{\partial \theta} \right)$

formal way to write ODEs

$f(z) = \begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix}$ $\quad g(z) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ $\quad$ or $f(z) = \cos\theta \frac{\partial}{\partial x} + \sin\theta \frac{\partial}{\partial y}$

$\dot{z} = f(z)v + g(z)\omega$ $\quad g(z) = \frac{\partial}{\partial \theta}$

$f(z), g(z) \in \mathcal{V}(M)$, with $[f, g] \neq 0$

computing Lie bracket

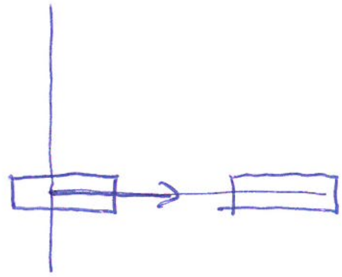
$$[f, g](z) = \frac{\partial g(z)}{\partial z} f(z) - \frac{\partial f(z)}{\partial z} g(z)$$

$$= 0 - \begin{bmatrix} 0 & 0 & -\sin\theta \\ 0 & 0 & \cos\theta \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

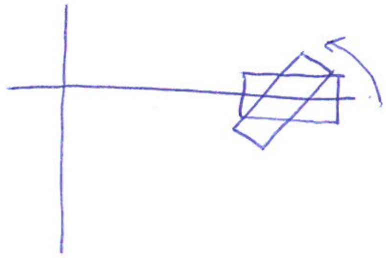
$$= \begin{bmatrix} \sin\theta \\ -\cos\theta \\ 0 \end{bmatrix} \triangleq h(z) \text{ new vector field}$$

measuring in terms of infinitesimal flow

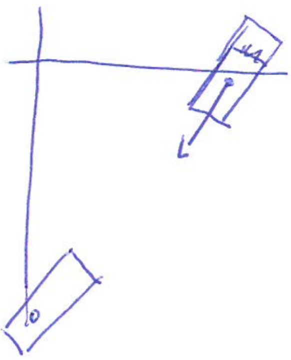
$\Phi^f(z_0)$



$\Phi^g \circ \Phi^f(z_0)$

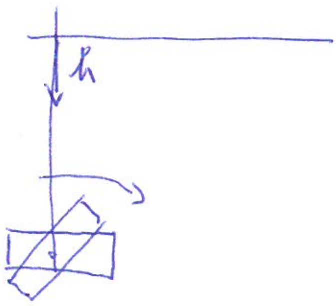


$\Phi^{-f} \circ \Phi^g \circ \Phi^f(z_0)$



$\Phi^{-g} \circ \Phi^{-f} \circ \Phi^g \circ \Phi^f(z_0)$

net motion



the direction of this net motion is represented by the v.f. $h(z)$

<u>example</u> linear switching system with commuting vector fields

$\dot{x} = f_i(x) \quad i \in P = \{1, 2\}, \quad f_i(x) = A_i x \quad A_i$ Hurwitz

Take any $\sigma(t)$

Assume $[f_1, f_2](x) = 0$ i.e. $[A_1, A_2] = 0$

i.e. $A_1 A_2 = A_2 A_1$

Flow at t arbitrary:

$$\Phi(x_0) = \Phi^{f_1}_{t_k} \circ \dots \circ \Phi^{f_1}_{t_3} \circ \Phi^{f_2}_{t_2} \circ \Phi^{f_1}_{t_1}(x_0) \text{ for } t_i \text{ arbitr.}$$

$$= e^{A_1 t_k} \cdot \dots \cdot e^{A_1 t_3} \cdot e^{A_2 t_2} e^{A_1 t_1} x_0$$

$\leftarrow$ true only because $[A_1 A_2] = 0$!!

$$= e^{(A_1 t_k + \dots + A_1 t_3 + A_2 t_2 + A_1 t_1)} x_0$$

since $\Phi^{f_1}_s \circ \Phi^{f_2}_t = \Phi^{f_2}_t \circ \Phi^{f_1}_s$ I can change the order of the terms arbitrarily

$$\Rightarrow \Phi(x_0) = e^{A_1 (\sum_k t_k)} e^{A_2 (\sum_j t_j)} x_0$$

as $t \to \infty$ at least one of the two summations $\to \infty$

but $e^{A_i t} \xrightarrow{t \to \infty} 0$ since A_i Hurwitz

$\Rightarrow x(t) \to 0 \quad \forall \sigma(t) \Rightarrow \bar{x} = 0$ unif. asympt. st.

Notice that $e^{A_1 t_1} e^{A_2 t_2} \neq e^{A_1 t_1 + A_2 t_2}$ in general

$$e^{A_1 t_1} e^{A_2 t_2} = e^{(A_1 t_1 + A_2 t_2 + [A_1, A_2] \frac{t_1 t_2}{2} + \dots)}$$ Campbell-Baker Hausdorff formula

n114

<u>example</u> ($so(3)$ composition) (linear v.f. on $so(3)$)

$$f(x) = A_3 x = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad g(x) = A_2 x = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} x$$

$A_i \in so(3)$ skew symm matrices

$$[f, g](x) = [A_2, A_3]x = A_1 x = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} x$$

flows along these v.f. are rotation matrices along the principal axes

$$e^{A_3 t} = \begin{bmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad e^{A_2 t} = \begin{bmatrix} \cos t & 0 & \sin t \\ 0 & 1 & 0 \\ -\sin t & 0 & \cos t \end{bmatrix}$$

apply the sequence of rotations

A_3 for $t = \pi/2$ sec (antraclockwise)

A_2 for $t = \pi/2$ sec "

$-A_3$ for $t = \pi/2$ sec clockwise

$$e^{-A_3 t} e^{A_2 t} e^{A_3 t} = e^{A_1 t} \qquad \text{for } t = \pi/2$$

i.e. combination of rotations along z and y axis produce rotation along x axis

(n115)

interpretation: if f and g commute

$\Phi^g_s \circ \Phi^f_t = \Phi^f_t \circ \Phi^g_s \Rightarrow \Phi^{-f}_t \circ \Phi^g_s \circ \Phi^f_t = \Phi^g_s$

in our case $\Phi^{-f} \circ \Phi^g \circ \Phi^f \neq \Phi^g$ hence $[f, g] \neq 0$

further interpretation: if f is complete, then

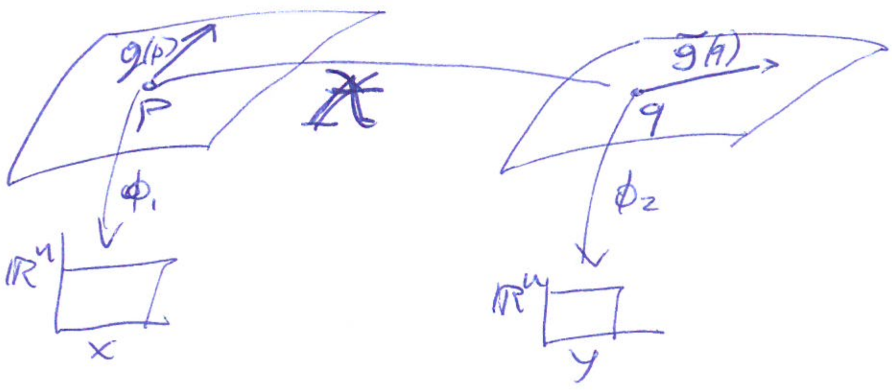
Φ^f_t form a one-param subgroup, and

$\Phi^{-f}_t = \left(\Phi^{+f}_t\right)^{-1} = \underbrace{\Phi^t_{-t}}_{?}$

hence $\left(\Phi^f_t\right)^{-1} \circ \Phi^g_s \circ \Phi^f_t$ can be seen as a

change of basis with $\left(\Phi^f_t\right)^{-1}$ as diffeomorphism

meaning on manifold:



maybe p and q on the same manifold

$\mathcal{H}$ = diffeomorphism $q = \mathcal{H}(p)$

if $g \in \mathcal{V}(M)$ then g defines ODE $\mathring{p} = f(p)$

in coord $x = \phi_1(p)$ $\quad \mathring{x} = f(x)$

complete map $\quad x=\phi_1(p) \Rightarrow p=\phi_1^{-1}(x)$

$\quad\quad y=\phi_2(q) \Rightarrow q=\phi_2^{-1}(y)$

$$\Rightarrow y=\phi_2(q)=\phi_2(\mathcal{X}(p))=\phi_2(\mathcal{X}(\phi_1^{-1}(x)))$$

for us $y \simeq \mathcal{X}(x)$ directly to shorten notations

from $y=\mathcal{X}(x)$

$$\dot{y}=\frac{\partial \mathcal{X}}{\partial x}\dot{x}=\frac{\partial \mathcal{X}}{\partial x}g(x)\Big|_{x=\mathcal{X}^{-1}(y)} \quad \text{ODE in } y \text{ (i.e. at } q\text{)}$$

$$=\tilde{g}(y)$$

this operator on manifolds is called "Ad"

$$Ad_{\mathcal{X}}\, g=\tilde{g}$$

local flow along $Ad_{\mathcal{X}}$: integrate ODE, or
use expression of the flow at p along $g(p)$

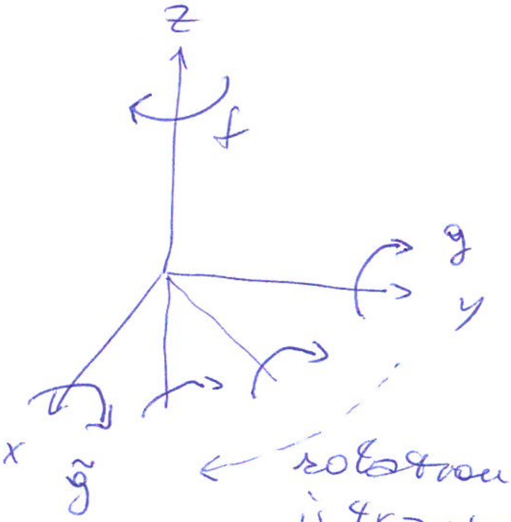
$$\Phi_s^{\tilde{g}}(q)=\mathcal{X}\circ\Phi_s^{g}(p)\circ\mathcal{X}^{-1}$$

If $\mathcal{X}$ is itself a flow along another v.f.
say f $\quad \mathcal{X}=\Phi_t^{-f}=\Phi_{-t}^{f}$ $\quad$ v.f. (g) changes as it is transported along the flow of another v.f.

then we have the case of the previous example -

this will be useful to study systems with drift

In the so(3) example: drift is $f(x)$ rotation around z and $g(x)$ (rotation around y) is transported along Φ^f until it becomes a rotation around x (i.e. $[f, g]$)



rotation around y (i.e. $g(x)$) is transported via the diffeom. Φ^f (flow along f = rotation around z), until it becomes a rotation around x

Distributions

We have seen vector fields $f \in \mathcal{V}(M)$ which at each $p \in M$ associate a tangent vector

$$f : M \longrightarrow TM$$
$$p \longmapsto f(p)$$

A distribution does the same thing for tangent subspaces

(here distribution is a term from differential geometry, nothing to do with probability distr.)

<u>def</u> A <u>distribution</u> Δ is a map assigning to $p \in M$ a vector subspace of T_pM
i.e. $\Delta(p) \subset T_pM$

$$\Delta : M \longrightarrow TM$$
$$p \longmapsto \Delta(p) \subset T_pM$$

Special case: $\Delta = \text{span}\{f_1, \dots f_k\}$ $f_i \in \mathcal{V}(M)$

(smooth distributions are always like this, at least locally)

$\Delta \subset TM$

span here is over functions, but leads to

$\Delta(p) = \text{span}\{f_1(p), \dots, f_k(p)\} \subset T_pM$ at any point $p \in M$

def A distribution is regular if $\dim \Delta(p) = k$ $\forall p \in M$

def A distribution Δ is involutive if $\forall f_i, f_j \in \Delta$ $[f_i, f_j] \in \Delta$ i.e. Δ is closed under the Lie bracket

def the involutive closure $\bar{\Delta}$ of Δ is the smallest involutive distribution containing Δ and closed w.r.t. the Lie bracket

$\Rightarrow$ $\bar{\Delta}$ is a Lie algebra (since it is closed w.r.t. Lie bracket)

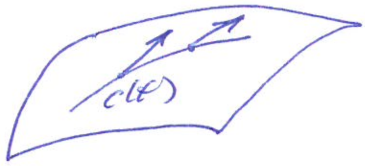
often indicated $\bar{\Delta} = \text{Lie}\{f_1, \dots, f_k\}$

(Lie algebra generated by v.f. $f_1, \dots, f_k$)

<u>def</u> A (embedded, immersed) submanifold $N \subset M$ is an <u>integral manifold</u> of a distribution Δ if $\Delta(p) = T_p N \ \forall p \in N$

<u>def</u> a distrib. Δ is said <u>integrable</u> if for any $p \in M$ passes an integrable manifold $N \subset M$ s.t. $\Delta(p) = T_p N$

<u>meaning</u>: for $\dot{c}(t) = f(c(t))$ an integral curve is the 1-dim integral manifold of the v.f. f, which is its infinitesimal generator



Here we want to do the same thing with a k-dim manifold and a k-dim distribution (integrable)

In order to do that, the v.f. $\{f_1, .., f_k\}$ which generate integrable Δ have to produce no new directions when we take Lie brackets

For integrable distrib: $\dim(N) = \dim(\Delta)$

this is the argument behind the so-called
Frobenius theorem

thm
A regular distribution Δ is integrable
$\Leftrightarrow$ Δ is involutive

Another interpretation:

Δ integrable of dim k if $\exists$ $n-k$ functions
$h_i : M \to \mathbb{R}$ s.t.

* $\frac{\partial h_i}{\partial x}$ lin. indep. (these are gradients)
$i = 1, \dots, n-k$ and form a rectangular Jacob. which has to have full rank

* $L_f h_i = \sum_j f_j \frac{\partial h_i}{\partial x_j} = \frac{\partial h_i}{\partial x} f(x) = 0$
$\forall f \in \Delta$ $\quad \forall i = 1, \dots, n-k$

then the submanifold N is an hypersurface given by the level sets of the functions h_i

$N = \{ p \in M \text{ s.t. } h_1(p) = c_1, \dots, h_{n-k}(p) = c_{n-k} \}$ $c_i = \text{const.}$

is an integral manifold of $\Delta = \text{span}\{f_1, \dots, f_k\}$

i.e. evolution driven by $f_1, \dots, f_k$, starting in $\mathcal{N}$ stays in $\mathcal{N}$ regardless of how we e.g. jump from f_i to f_j, or of how we follow a vector field which is a linear combination of the f_i (i.e. $f = \sum \alpha_i f_i$)

Here $\mathcal{N}$ is $\mathcal{N}_{c_1, \dots, c_{n-k}}$ i.e. it is one level surface of a foliation, that is M is foliated into level surfaces which are submanifold $\mathcal{N}_{c_1, \dots, c_{n-k}}$ as we vary the constants c_i : $M = \bigcup_{c_1 \dots c_{n-k}} \mathcal{N}_{c_1, \dots, c_{n-k}}$

<u>example</u>: rolling wheel

$$f(x) = \begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix} \qquad g(x) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad \Delta = \mathrm{span}\{f, g\}$$

$$[f, g] = \frac{\partial g}{\partial x} f - \frac{\partial f}{\partial x} g = \begin{bmatrix} \sin\theta \\ -\cos\theta \\ 0 \end{bmatrix} \triangleq \ell(x) \in \mathfrak{X}(M)$$

new v.f.

We want to check involutivity taking more Lie brackets

$$[\ell, f], \ [\ell, g] \in \bar{\Delta} = \mathrm{span}\{f, g, \ell\} = TM$$

i.e. $\dim(\bar{\Delta}) = \dim(T_pM)$ and it cannot grow further

$\Rightarrow$ $\bar{\Delta}$ is involutive

$\Rightarrow$ smallest integrable distrib. is the entire tangent bundle.

$\Rightarrow$ there are no 2-dim integrable submanf. in $M = \mathbb{R}^2 \times S$ if we start from $\Delta = \mathrm{span}\{f, g\}$

example linear v.f. on $so(3)$

$$f(x) = A_1 x = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} x \qquad g(x) = A_2 x = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} x$$

$A_i \in so(3)$

$$[f, g](x) = (A_2 A_1 - A_1 A_2) x = [A_2, A_1] x = -A_3 x$$

with $A_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ lin indep. from A_1, A_2

(in fact $so(3) = \text{span}\{A_1, A_2, A_3\}$ is a matrix Lie algebra of dim 3)

$\Rightarrow$ matrices A_1 and A_2 do not commute.

However, if I look at the v.f.

$$f(x) = \begin{bmatrix} 0 \\ -x_3 \\ x_2 \end{bmatrix} \qquad g(x) = \begin{bmatrix} x_3 \\ 0 \\ -x_1 \end{bmatrix} \qquad \ell(x) = \begin{bmatrix} -x_2 \\ x_1 \\ 0 \end{bmatrix}$$

For $x_i \neq 0$ initial distrib is

$$\Delta = \text{span}\{f, g\} \qquad \Delta(x) = \text{span}\left\{ \begin{bmatrix} 0 \\ -x_3 \\ x_2 \end{bmatrix}, \begin{bmatrix} x_3 \\ 0 \\ -x_1 \end{bmatrix} \right\}$$

which has dim $= 2$

while $\text{span}\{f(x), g(x), \ell(x)\} = \text{span}\left\{\begin{bmatrix} 0 \\ -x_3 \\ x_2 \end{bmatrix}, \begin{bmatrix} x_3 \\ 0 \\ -x_1 \end{bmatrix}, \begin{bmatrix} -x_2 \\ x_1 \\ 0 \end{bmatrix}\right\}$

has still dim = 2 !

to see it, taking $\alpha_1 = \frac{x_1}{x_3}$, $\alpha_2 = -\frac{x_2}{x_3}$, $\ell(x)$ is a function of $f(x)$, $g(x)$

$$\ell(x) = \begin{bmatrix} -x_2 \\ x_1 \\ 0 \end{bmatrix} = \alpha_1 f(x) + \alpha_2 g(x) = \begin{bmatrix} 0 \\ -x_1 \\ -\frac{x_1 x_2}{x_3} \end{bmatrix} + \begin{bmatrix} -x_2 \\ 0 \\ +\frac{x_1 x_2}{x_3} \end{bmatrix}$$

$\Rightarrow$ distribution Δ we start from is already involutive (out of singular points like $x_3 = 0$)

$\Rightarrow$ (Frobenius theorem) Δ is integrable

$\Rightarrow$ there must exist an integral manifold $\mathcal{U}$ of dim 2 - How to find it ?

To find $\mathcal{U}$ we have to determine $n-k=1$ function $h: M \to \mathbb{R}$ s.t.

* $\frac{\partial h}{\partial x} \neq 0$ (trivial)
* $L_f h(x) = 0$ and $L_g h(x) = 0$

Try with $h(x) = \|x\|_2^2 = x_1^2 + x_2^2 + x_3^2$

$$\frac{\partial h}{\partial x} = 2(x_1 + x_2 + x_3) \neq 0 \quad \text{in } x \neq 0$$

$$L_f h(x) = \frac{\partial h}{\partial x} f(x) = 2\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 0 \\ -x_3 \\ x_2 \end{bmatrix} =$$

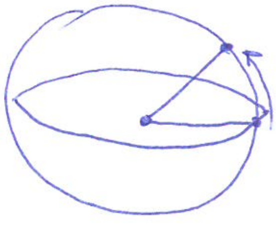
$$= 2(-x_2 x_3 + x_2 x_3) = 0$$

$$L_g h(x) = \frac{\partial h}{\partial x} g(x) = 2\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} x_3 \\ 0 \\ -x_1 \end{bmatrix} =$$

$$= 2(x_1 x_3 - x_3 x_1) = 0$$

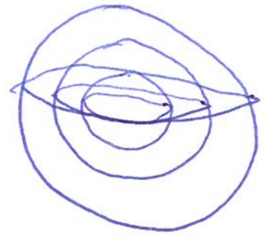
$\Rightarrow$ we have found an integral manifold of $\Delta = \text{span}\{f, g\}$, and it is S^2!

meaning: $f(x)$ and $g(x)$ are infinitesimal generators of rotations, i.e. a vector $x \in \mathbb{R}^3$ is rotated but its length is preserved when it flows along $f(x)$ or $g(x)$!



$$h(x) = \|x\|_2^2 = c \overset{!}{=} \text{const} \quad (= \text{radius})$$

induces a foliation in $\mathbb{R}^3$ with concentric spheres as leaves



Controllability

Consider an autonomous control-affine nonlinear system

$$(*) \quad \dot{x} = f(x) + \sum_{i=1}^{m} g_i(x) u_i \qquad x \in M$$

(more properly $x = \phi(p)$ local coord. $p \in M$)

u_i = controls $m < n$ (underactuated system)

$u_i \in \mathcal{U}$ class of piecewise-const. functions (approximates well any class of control inputs of interest)

$u_i : [0, T] \to U \subset \mathbb{R}$ U values admissible for the control (depends on the problem) e.g. $U = [-1, 0, 1]$, $U = \mathbb{R}$, etc.

$f \in \mathcal{V}(M)$, = drift v.f.

$g_i \in \mathcal{V}(M)$ = input v.f.

<u>def</u> the system $(*)$ is <u>controllable</u> if $\forall x_1, x_2 \in M$ $\exists T$ (finite) and control functions $u_i : [0, T] \to U$, $u_i \in \mathcal{U}$ s.t. $x(0) = x_1$

$x(T) = x_2$