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Chained form

A chained form is a canonical form for rank-2 distributions having growth vector $r = \{2, 3, 4, 5, \dots n\}$

$$\dot{x}_1 = u_1$$
$$\dot{x}_2 = u_2$$
$$\dot{x}_3 = x_2 u_1$$
$$\dot{x}_4 = x_3 u_1$$
$$\vdots$$
$$\dot{x}_n = x_{n-1} u_1$$

$$\Rightarrow \dot{x} = \begin{bmatrix} 1 \\ 0 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} u_2 = g_1(x) u_1 + g_2(x) u_2$$

degree of nonholonomy is $n-2$. To show: compute the filtration

$\Delta_1 = \text{span}\{g_1, g_2\}$ has rank 2

$$[g_1, g_2] = 0 - \frac{\partial g_1}{\partial x} g_2 = - \begin{bmatrix} 0 & \cdots & & 0 \\ 0 & \cdots & & 0 \\ 0 & 1 & 0 \cdots & 0 \\ 0 & 0 & 1 \cdots & 0 \\ & & \ddots & \\ 0 & \cdots & 0 \; 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = g_3$$

$\Delta_2 = \Delta_1 + [\Delta_1, \Delta_1] = \text{span}\{g_1, g_2, g_3\}$ has rank 3

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2nd level Lie bracket:

$$[g_1, g_3] = 0 - \frac{\partial g_1}{\partial x} g_3 = \begin{bmatrix} 0 & \cdots & & 0 \\ 0 & \cdots & & 0 \\ 0 & 1 & \cdots & 0 \\ 0 & 0 & 1 & \\ & & \ddots & \\ & & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = g_4$$

$\Rightarrow \Delta_3 = \Delta_2 + [\Delta_1, \Delta_2] = \mathrm{span}\{g_1, g_2, g_3, g_4\} \Rightarrow$ has ~~dim~~ rank 4

$\Rightarrow \Delta_{n-1} = T_x \mathbb{R}^n = \bar{\Delta} \quad \Rightarrow$ LARC $\Rightarrow$ STCL

$\Rightarrow r = \{2, 3, 4, \dots, n\}$

In addition: only $n-2$ Lie brackets are $\neq 0$ i.e. the minimal number of Lie brackets needed to achieve LARC

all the Lie brackets are of the form

$[g_1, [g_1, \dots, [g_1, g_2] \dots]] = ad^k_{g_1} g_2$

"as in a linear system" (the system is bilinear)

Since $\underbrace{[g_1, [g_1, \dots, [g_1, g_2] \dots]]}_{n\text{-times}} = ad^n_{g_1} g_2 = 0$

all n-level Lie brackets are 0, and so are all higher level Lie bracket

$\rightarrow$ nilpotent system

<u>example</u> N-trailer system

has growth vector $r = \{2, 3, 4, \dots, n\}$

$\Rightarrow$ it is feedback equival. to a chained form

i.e. $\exists$ $\begin{cases} z = \Phi(x) & \text{diffeom.} \\ v_i = \varphi_i(x, u_i) & \text{diffeom in } u_i \end{cases}$

s.t. N-trailer $\dot{x} = \sum g_i(x) u_i$

can be transf. into a chained form by

$$\dot{z} = \sum_{i=1}^{2} \left.\frac{\partial \phi}{\partial x}\right|_{x=\phi^{-1}(z)} g_i\left(\Phi^{-1}(z)\right) \varphi_i\left(\Phi^{-1}(z), \varphi^{-1}(u_i)\right)$$

(I do not need to do the explicit calcul!)

<u>example</u> : rolling wheel

$$\dot{z} = \begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix} v + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} w$$

growth vector $r = \{2, 3\}$

$\Rightarrow$ feedback equivalent to a chained form

$$\begin{cases} \dot{x}_1 = u_1 \\ \dot{x}_2 = u_2 \\ \dot{x}_3 = x_2 u_1 \end{cases}$$

(original state : z
chained form : x)

computing explicitly:

choose : $\cancel{X_1} (= x) \triangleq z_1$

$\cancel{X_3} (= y) \triangleq z_2$

$u_1 \triangleq v\cos\theta$

$x_2 \triangleq \tan\theta$

$\Uparrow$

hence $\cancel{\dot{z}_1} = \cancel{\dot{X}_1} = v\cos\theta \triangleq u_1$

$$\dot{z}_2 = \dot{X}_3 = v\sin\theta = \frac{u_1 \sin\theta}{\cos\theta} = u_1 \tan\theta \triangleq u_1 x_2$$

derive x_2 : $\dot{x}_2 = \frac{\dot{\theta}}{\cos^2\theta} = \frac{w}{\cos^2\theta} \triangleq u_2$

$\Rightarrow$ complete transformation is

$$\begin{cases} x_1 = z_1 \\ x_2 = \tan z_3 \\ x_3 = z_2 \end{cases} \qquad \begin{cases} u_1 = v \cos z_3 \\ u_2 = \dfrac{w}{\cos^2 z_3} \end{cases}$$

which is a feedback equivalence.

example : car-like vehicle

$$\begin{cases} \dot{x} = v\cos\theta \\ \dot{y} = v\sin\theta \\ \dot{\theta} = v\dfrac{\tan\beta}{L} \\ \dot{\beta} = w \end{cases} \quad \text{in } z \text{ variables} \quad \begin{aligned} \dot{z}_1 &= v\cos z_3 \\ \dot{z}_2 &= v\sin z_3 \\ \dot{z}_3 &= v\frac{\tan z_4}{L} \\ \dot{z}_4 &= w \end{aligned}$$

change into chained form : $\begin{cases} \dot{x}_1 = u_1 \\ \dot{x}_2 = u_2 \\ \dot{x}_3 = x_2 u_1 \\ \dot{x}_4 = x_3 u_1 \end{cases}$

choose $u_1 \triangleq v \cos z_3$

$x_1 \triangleq z_1$

$x_4 \triangleq z_2$

derive : $\dot{x}_4 = \dot{z}_2 = v\sin z_3 = u_1 \frac{\sin z_3}{\cos z_3} = u_1 \underbrace{\tan z_3}_{\triangleq x_3}$

i.e. $x_3 \triangleq \tan z_3$

derive

$$\dot{x}_3 = \frac{d}{dt}\tan z_3 = \frac{\dot{z}_3}{\cos^2 z_3} = v\frac{\tan z_4}{L\cos^2 z_3} \stackrel{*}{=} u_1 \frac{\tan z_4}{L\cos^3 z_3}$$

define: $x_2 \triangleq \frac{\tan z_4}{L\cos^3 z_3}$

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derive x_2: $\dot{x}_2 = \frac{d}{dt}\left(\frac{\tan z_4}{L\cos^3 z_3}\right) =$

$$= \frac{\frac{L\cos^{\cancel{3}1} z_3}{\cos^2 z_4}\dot{z}_4 - 3L\cancel{\cos^2} z_3 \tan z_4 \dot{z}_3(-\sin z_3)}{L^2 \cos^{\cancel{6}4} z_3}$$

$$= \frac{L\frac{\cos z_3 W}{\cos^2 z_4} + 3\cancel{L}\sin z_3 \frac{\tan^2 z_4}{\cancel{L}}\frac{u_1}{\cos z_3}}{L^2\cos^4 z_3}$$

$$= \frac{L\cos^2 z_3 W + 3\sin z_3 \sin^2 z_4 u_1}{L^2\cos^2 z_4 \cos^5 z_3} \hat{=} u_2$$

complete transf. is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} z_1 \\ \frac{\tan z_4}{L\cos^3 z_3} \\ \tan z_3 \\ z_2 \end{bmatrix} \qquad \left\{ \begin{array}{l} u_1 = v\cos z_3 \\ u_2 = \frac{W}{L\cos^3 z_3\cos^2 z_4} + 3\frac{\sin^2 z_3}{\cos^4 z_3}\tan^2 z_4 v \end{array} \right.$$

Motion planning for driftless systems

Consider the system

$$\dot{x} = \sum_{i=1}^{m} g_i(x) u_i$$

Motion planning problem:

Given x_0 and x_f, how to choose $u_i \in \mathcal{U}$ s.t.
$x(0) = x_0$ and $x(T) = x_f$ for some $T > 0$?

- for linear systems: optimal control, syst. invers.
  Model Pred. Control

- for nonlinear systems: very difficult problem
  - open-loop control (to generate a reference trajectory)
  - then (if possible): feedback stabiliz around the open-loop trajectory (or trajectory tracking)

- Today: examples of methods for open-loop control for special cases of driftless nonlinear control systems.

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Optimal control of driftless systems

consider $\dot{x} = \sum_{i=1}^{m} g_i(x) u_i$ under sat. $m < n$ controll (STLC)

want to steer the state from x_0 to x_f in time 1

$x(0) = x_0 \qquad x(1) = x_f$

while minimizing the control energy

$$\frac{1}{2}\int_0^1 \|u(\tau)\|^2 d\tau = \frac{1}{2}\int_0^t u^T(\tau)u(\tau)\, d\tau$$

- Here: only heuristic derivation of the necessary conditions from calculus of variations
- construct <u>cost function J</u> which includes the constraints as Lagrange multipliers $\lambda(t)$

$$J(x,\lambda,u) = \int_0^1 \left(\frac{1}{2} u^T(\tau)u(\tau) - \lambda^T\left(\dot{x} - \sum_{i=1}^{m} g_i(x)u_i\right)\right) d\tau$$

- Introduce the <u>Hamiltonian function</u>

$$H(x,\lambda,u) \triangleq \frac{1}{2}u^T u + \lambda^T \sum_{i=1}^{m} g_i(x) u_i$$

using this and integrating by part the $\lambda^T \dot{x}$ term of J

$$J(x,\lambda,u) = -\lambda^T(t)x(t)\Big]_0^1 + \int_0^1 \left(H(x,\lambda,u) + \dot{\lambda}^T x\right) d\tau$$

where $\frac{d}{dt}(\lambda^T x) = \dot{\lambda}^T x + \lambda^T \dot{x} \Rightarrow -\int\left(\frac{d}{dt}(\lambda^T x) + \dot{\lambda}^T x\right) = \int -\lambda^T \dot{x}$

use calculus of variations around the optimum (m.sector)
Consider variation in u: δu and the variation in x they induce: δx.
These induce variations in the cost J: δJ

$$\delta J = -\lambda^T(t)\delta x(t)\Big]_0^1 + \int_0^1 \left(\frac{\partial H}{\partial x}^T \delta x + \frac{\partial H}{\partial u}^T \delta u + \dot{\lambda}^T \delta x \right) d\tau$$

If optimum has been found a necessary cond is that $\delta J \equiv 0$ $\forall$ variations δu and δx
i.e.

$$\begin{cases} \dot{\lambda} = -\dfrac{\partial H}{\partial x}^T \\ \dfrac{\partial H}{\partial u} = 0 \end{cases}$$

(i.e. J has a minimum $\Rightarrow \delta J = 0$)
<u>Euler-Lagrange eq.</u>

From the second it follows:
$u_i = -\lambda^T g_i(x)$ $i = 1 \dots m$
(optimal controls expressed as function of adjoint variables λ)

$\Rightarrow$ optimal Hamiltonian is then

$$H^*(x, \lambda) = -\frac{1}{2} \sum_{i=1}^m \left(\lambda^T g_i(x) \right)^2$$

From which one gets the <u>Hamiltonian eq. for the optimal control problem</u>

$$\begin{cases} \dot{x} = \dfrac{\partial H^{*T}}{\partial \lambda}(x, \lambda) = -\sum_i g_i(x) \left(\lambda^T g_i(x) \right) \\ \dot{\lambda} = -\dfrac{\partial H^{*T}}{\partial x}(x, \lambda) = \sum_i \dfrac{\partial g_i}{\partial x}^T \lambda \left(\lambda^T g_i(x) \right) \end{cases}$$

with boundary cond $x(0) = x_0$, $x(1) = x_f$
(or $\lambda(1) = 0$)

- Problem: difficult to solve / use in practice
  - necessary cond (not suff)
  - abnormal minimizers (because of the "missing directions" $\rightarrow$ sub-Riemannian geom.)

Steering chained-form systems using sinusoids

$\dot{x}_1 = u_1$
$\dot{x}_2 = u_2$
$\dot{x}_3 = x_2 u_1$
$\vdots$
$\dot{x}_n = x_{n-1} u_1$

want to steer x from x_0 to x_f

idea: use sinusoidals at integrally related frequencies

ex: if $\begin{cases} u_1 = \sin 2\pi t \\ u_2 = \cos 2\pi k t \end{cases} \Rightarrow$
$\dot{x}_3$ has component at freq. $2\pi(k-1)$
$\dot{x}_4$ " " " " $2\pi(k-2)$
$\vdots$
$\dot{x}_{k+2}$ " " " " 0

$\Rightarrow$ when integrated between 0 and 1 $x_1, x_2, \dots x_{k+1}$ return to their initial value, while x_{k+2} has a neat motion (frequency 0 $\Rightarrow$ a constant)

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Algorithm: $(t_0 = 0\ ,\ t_{final} = 1)$

1) steer x_1 and x_2 to their desired values (using proper u_1 and u_2) ($\dot{x}_1 = u_1$, $\dot{x}_2 = u_2$ are both first order integrators $\Rightarrow$ trivial)

2) for each x_{k+2}, $k \geq 1$, (i.e. $x_3, x_4 \dots$) steer x_{k+2} to its final value using $u_1 = a \sin 2\pi t$, $u_2 = b \cos 2k\pi t$ where a and b s.t. $x_{k+2}(1) - x_{k+2}(0) = \left(\frac{a}{4\pi}\right)^k \frac{b}{k!}$

proof (idea)

Why x_i $i = 1, \dots, k+1$ are such that $x_i(1) = x_i(0)$ with these inputs?

Use trigonometric formula

$$\int \sin \alpha\tau \sin \beta\tau \, d\tau = \frac{1}{2}\int\left(\cos(\alpha-\beta)\tau - \cos(\alpha+\beta)\tau\right)d\tau = \underbrace{\frac{\sin(\alpha-\beta)\tau}{2(\alpha-\beta)}}_{\text{term at lower freq.}} - \underbrace{\frac{\sin(\alpha+\beta)\tau}{2(\alpha+\beta)}}_{\text{term at higher freq.}}$$

Using this formula, with u_1 and u_2 given above

$$x_1 = a\int_0^1 \sin 2\pi\tau \, d\tau = \frac{a}{2\pi}\Big[-\cos 2\pi\tau\Big]_0^1 = -\frac{a}{2\pi}\left[\cos 2\pi - 1\right] = 0$$

$$x_2 = b\int_0^t \cos 2\pi k\tau \, d\tau = +\frac{b}{2\pi k}\sin 2\pi k\tau\Big]_0^t = +\frac{b}{2\pi k}\sin 2\pi k t$$

use this inside $\dot{x}_3 = x_2 u_1$

$$\dot{x}_3 = x_2 u_1 = +\frac{ab}{2k\pi}\sin 2\pi k t \sin 2\pi t$$

now integrate this

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$$x_3(t) = +\frac{ab}{2k\pi}\int_0^t \sin 2k\pi\tau \sin 2\pi\tau \, d\tau$$

$$= +\frac{ab}{2k\pi}\left[\underbrace{\frac{\sin((k-1)2\pi\tau)}{2\pi(k-1)}}_{\text{only this matter}} - \frac{\sin((k+1)2\pi\tau)}{2\pi(k+1)}\right]_0^t$$

similarly

$$\dot{x}_4 = x_3 u_1 = \left(-\frac{ab}{2k\pi}\frac{\sin((k-1)2\pi t)}{2\pi(k-1)} + \dots\right) a \sin 2\pi t$$

integrating

$$x_4(t) = \frac{1}{2}\frac{a^2 b}{2\pi k \cdot 2\pi(k-1)}\int_0^t \sin 2\pi(k-1)\tau \sin 2\pi\tau \, d\tau + \dots$$

$$= \frac{1}{2}\frac{a^2 b}{(2\pi)^3 k \cdot (k-1)(k-2)} \sin((k-2)2\pi t) + \dots$$

$\vdots$

$$x_{k+2} = \frac{1}{2^{k-1}}\frac{a^k b}{(2\pi)^k k!}\int_0^t \overbrace{\sin 2\pi\tau \cdot \sin 2\pi\tau \, d\tau}$$

$$= \int_0^t \sin^2 2\pi\tau \, d\tau \neq 0$$ when integrated between 0 and 1

Motion planning via differential flatness

Consider a nonlinear control system

$\dot{x} = f(x,u)$ underactuated $m < n$

<u>def</u> the system is <u>differentially flat</u> if $\exists$ m (= n° of inputs) "output" functions $y_1(t), \dots, y_m(t)$ (call $y \triangleq \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix}$) s.t. both x and u can be expressed as algebraic functions of y and of its derivatives i.e.

$$x = \phi(y, \dot{y}, \ddot{y}, y^{(3)}, \dots)$$
$$u = \psi(y, \dot{y}, \ddot{y}, y^{(3)}, \dots)$$

- y is called <u>flat output</u> (vector of flat outputs)
- differential flatness is a form of <u>system inversion</u>: the whole system (state and input functions) becomes a trivial algebraic expression of y.

$\Rightarrow$ motion planning problem can be solved exactly by specifying $y(t)$.

<u>example</u> chained form

$$\begin{cases} \dot{x}_1 = u_1 \\ \dot{x}_2 = u_2 \\ \dot{x}_3 = x_2 u_1 \\ \vdots \\ \dot{x}_n = x_{n-1} u_1 \end{cases}$$

n states

$m = 2$ inputs $\Rightarrow$ 2 flat outputs

choose as flat outputs

$$y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_n \\ x_1 \end{bmatrix}$$

deriving $\dot{y} = \begin{bmatrix} \dot{x}_n \\ \dot{x}_1 \end{bmatrix} = \begin{bmatrix} x_{n-1} u_1 \\ u_1 \end{bmatrix}$

$\Rightarrow u_1 \triangleq \dot{y}_2$ (I have obtained an input as function of the flat outputs)

$\Rightarrow x_{n-1} = \dfrac{\dot{y}_1}{u_1} \triangleq \dfrac{\dot{y}_1}{\dot{y}_2}$ (I have obtained x_{n-1} as function of the flat outp.)

derive again y_1:

$$\ddot{y}_1 = \dot{x}_{n-1} u_1 + x_{n-1} \dot{u}_1 = x_{n-2} u_1^2 + \frac{\dot{y}_1}{\dot{y}_2} \ddot{y}_2$$

$\Rightarrow$ can obtain x_{n-2}

$$x_{n-2} = \frac{\left(\ddot{y}_1 - \dfrac{\ddot{y}_2 \dot{y}_1}{\dot{y}_2} \right)}{u_1^2} = \frac{\ddot{y}_1}{\dot{y}_2^2} - \frac{\dot{y}_1 \ddot{y}_2}{\dot{y}_2^3} \triangleq \phi_{n-2}(y, \dot{y}, \ddot{y})$$

iterate the procedure (differentiate and extract x_{n-3} ...)

$\vdots$

$x_1 = \phi_1(y, \dot{y}, \ddot{y}, \dots)$

derive, and get

$u_2 = \psi_2(y, \dot{y}, \ddot{y}, \dots)$

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singularities of the procedure: terms at the denominator may vanish ⇒ must have $u_1 \neq 0$ i.e. $\dot{y}_2 \neq 0$ etc..
(these can be imposed when choosing a trajectory $y(t)$ on which to move the system).

- differential flatness can be used in conjunction with feedback equivalence

<u>~~example~~ remark</u>: if a system is feedback equivalent to a chained form then it is also differentially flat.

<u>example</u> rolling wheel

$$\begin{cases} \dot{x} = v\cos\theta \\ \dot{y} = v\sin\theta \\ \dot{\theta} = w \end{cases} \qquad \begin{aligned} n &= 3 \\ m &= 2 \end{aligned}$$

we know this is equivalent to
$$\begin{aligned} \dot{x}_1 &= u_1 \\ \dot{x}_2 &= u_2 \\ \dot{x}_3 &= x_2 u_1 \end{aligned}$$

⇒ we can take the flat outputs of the chained form and transform into flat outputs of the rolling wheel.

alternatively: we can compute them directly in the rolling wheel basis.

Choose $\begin{cases} y_1 = x \\ y_2 = y \end{cases}$ as flat outputs

derive

$$\dot{y}_1 = \dot{x} = v\cos\theta$$
$$\dot{y}_2 = \dot{y} = v\sin\theta \quad \Rightarrow \quad \frac{\dot{y}_2}{\dot{y}_1} = \frac{\dot{y}}{\dot{x}} = \tan\theta$$

$$\Rightarrow \theta \triangleq \operatorname{atan}\left(\frac{\dot{y}_2}{\dot{y}_1}\right)$$

observe that $\dot{x}^2 + \dot{y}^2 = v^2(\cos^2\theta + \sin^2\theta) = v^2$

$$\Rightarrow v \triangleq \sqrt{\dot{x}^2 + \dot{y}^2} = \sqrt{\dot{y}_1^2 + \dot{y}_2^2}$$

derive θ:

$$\dot{\theta} = w \triangleq \frac{d}{dt}\left(\operatorname{atan}\frac{\dot{y}_2}{\dot{y}_1}\right) = \frac{1}{1 + \left(\frac{\dot{y}_2}{\dot{y}_1}\right)^2}\frac{d}{dt}\left(\frac{\dot{y}_2}{\dot{y}_1}\right)$$

$$= \frac{\cancel{\dot{y}_1^2}}{\dot{y}_1^2 + \dot{y}_2^2}\,\frac{\ddot{y}_2\dot{y}_1 - \ddot{y}_1\dot{y}_2}{\cancel{\dot{y}_1^2}} = \frac{\dot{y}_1\ddot{y}_2 - \ddot{y}_1\dot{y}_2}{\dot{y}_1^2 + \dot{y}_2^2}$$

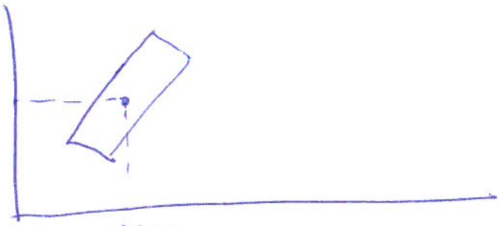
$\Rightarrow x, y, \theta$ and v, w are all expressed as functions of $y, \dot{y}, \ddot{y}, \ldots$

$\Rightarrow$ system is indeed differentially flat

As can be seen, the explicit calculation of flat outputs gets quickly messy, but it is nevertheless a very useful method to do motion planning, since the flat output coincide with the (x,y) coordinates of the rolling wheel.

Doing motion planning i.e. choosing x, y at $t=0$ and $v(t)$, $w(t)$ so that $x(t)$ and $y(t)$ are those given then becomes choosing a curve in $\mathbb{R}^2$ (e.g. a spline) between $x(0), y(0)$ and $x(T), y(T)$

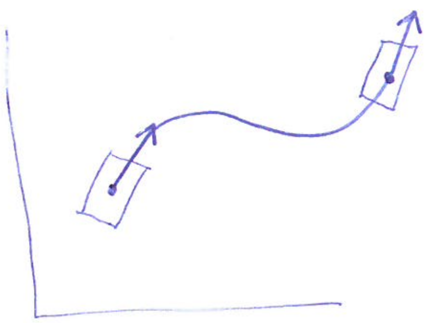
$y(0)$



$x(0)$

assigning also $\dot{x}(0), \dot{y}(0)$ and $\dot{x}(T), \dot{y}(T)$ means choosing a planar cubic spline

to respect the nonholonomic constraints, it is necessary to assign also θ (which is a function of y_1, y_2 hence it is available).



$x(0), y(0), \dot{x}(0), \dot{y}(0)$ and $x(T), y(T), \dot{x}(T), \dot{y}(T)$ can all be expressed as funct. of flat outputs $\Rightarrow$ $v(t)$ and $w(t)$ that allows to follow exactly that traj can be computed