PART 3: FEEDBACK LINEARIZATION

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Consider a control-affine nonlin syst

(*) $\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x) u_i$

and x_0 s.t. $f(x_0) = 0$

Control problem: find a feedback law
$u(t) = \gamma(x(t))$ s.t. x_0 is (locally) asympt. stable

$u(t) = \gamma(x(t))$ = static state feedback

What kind of approach can we follow.

1) Jacobian linearization around x_0

$A = \frac{\partial f(x_0)}{\partial x}$ $\quad B = [g_1(x_0) \dots g_m(x_0)]$

- If $\dot{x} = Ax + Bu$ is controllable
  $\Rightarrow$ choosing $u = Kx$ we can get
  $Re[\lambda(A+BK)] < 0$ i.e. locally stabilize the nonlinear system
- If $\dot{x} = Ax + Bu$ stabilizable (i.e. non-controllable modes have real part < 0) $\Rightarrow u = Kx$ can locally stabilize nonlinear system also in this case

$A \to \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix}$

$B \to \begin{bmatrix} 0 \\ B_2 \end{bmatrix}$

$Re[\lambda(A_{11})] < 0$

- advantage: simple! (n166)
- disadvantage: local result
  sufficient but not necessary
- mathematically not interesting ...

2) Lyapunov-based design
- use Lyapunov theory to construct control Lyapunov functions
- disadvantage: must find Lyapunov function!

3) today and next time: techniques based on exact linearization - 3 versions:
   i) state linearization = exact linear. by change of state variables
   ii) feedback linearization = exact linearization by change of state var and change of input (-> feedback equival)
   iii) input-output linearization: partial linearizat. of the dynamics via feedback equivalence

State linearization

= linearization by means of a change of state

def (*) is linearizable by means of a change of state if $\exists$ diffeomorphism $z = \Phi(x)$ s.t. $\Phi(0) = 0$ that transforms (*) into

$$\dot{z} = Ax + Bu$$

- Condition for this to happen

$$z = \Phi(x) \Rightarrow \dot{z} = \frac{\partial \Phi}{\partial x}\dot{x} = \frac{\partial \Phi}{\partial x}\left(f(x) + \sum g_i(x) u_i\right)$$

$$\overset{!}{=} Az + \sum b_i u_i \quad \left(B = [b_1 \dots b_m]\right)$$

$$\Leftrightarrow \begin{cases} \dfrac{\partial \Phi(x)}{\partial x} f(x) = A\,\Phi(x) \\ \dfrac{\partial \Phi(x)}{\partial x} g_i(x) = b_i \end{cases}$$

set of PDEs that must be solved

- Earlier we use the notation

$$\frac{\partial \Phi}{\partial x} f(x)\Big|_{x = \Phi^{-1}(z)} = Az \quad \left(\text{and also } Ad_{\Phi(x)} f\right)$$

which is the same thing.

• Necessary and sufficient "geometric" conditions for state linearization

<u>thm</u> the system (*) can be linearized to a controllable linear system at x_0 by means of a change of state iff near x_0 we have:

1) $\dim\left(\text{span}\{ad_f^k g_i(x),\ i=1,\dots,m,\ k=0,\dots n-1\}\right) = n$
$\forall x \in B(x_0)$

2) $[ad_f^k g_i, ad_f^\ell g_j] = 0 \quad \forall i,j=1,\dots,m, \ \forall k,\ell \geq 0$
$\forall x \in B(x_0)$

Conditions are similar to STLC we saw earlier - (case in which LARC implies STLC)

<u>Proof</u> (idea)

Assume $\exists\ \Phi$ s.t. $\left.\frac{\partial \Phi}{\partial x} f\right|_{x=\Phi^{-1}(z)} = Az$

$$\left.\frac{\partial \Phi}{\partial x} g_i\right|_{x=\Phi^{-1}(z)} = b_i$$

for some (A,B) controllable -

then $[f, g_i]$ is transformed to

$$\left.\frac{\partial \Phi}{\partial x}[f, g_i]\right|_{x=\Phi^{-1}(z)} = \left.\left[\frac{\partial \Phi}{\partial x} f, \frac{\partial \Phi}{\partial x} g_i\right]\right|_{x=\Phi^{-1}(z)} = [Az, b_i]$$

$$= -Ab_i \quad \text{(we computed this earlier)}$$

and similarly: $\frac{\partial \Phi}{\partial x} ad_f^k g_i \Big|_{x=\Phi^{-1}(0)} = (-1)^k A^k b_i$ (n169)

$\Rightarrow$ in order to have (A, B) controllable, cond 1) must hold

As for cond. 2):

$$\frac{\partial \Phi}{\partial x}[ad_f^k g_i, ad_f^\ell g_j]\Big|_{x=\Phi^{-1}(z)} = [(-1)^k A^k b_i, (-1)^\ell A^\ell b_j] = 0$$

since they are all const. v.f. //

<u>example</u> $x \in \mathbb{R}^2$

$$\dot{x}_1 = x_1 \ln x_2$$
$$\dot{x}_2 = -x_2 \ln x_1 + x_2 u \qquad x_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$f(x) = \begin{bmatrix} x_1 \ln x_2 \\ -x_2 \ln x_1 \end{bmatrix} \quad g(x) = \begin{bmatrix} 0 \\ x_2 \end{bmatrix}$$

$f(x_0) = 0$ ~~($\Rightarrow$ dim(span$\{f(x_0), g(x_0)\}$) = 1)~~

$$[f, g] = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \ln x_2 \\ -x_2 \ln x_1 \end{bmatrix} - \begin{bmatrix} \ln x_2 & \frac{x_1}{x_2} \\ -\frac{x_2}{x_1} & \ln x_1 \end{bmatrix}\begin{bmatrix} 0 \\ x_2 \end{bmatrix} =$$

$$= \begin{bmatrix} 0 \\ -x_2 \ln x_1 \end{bmatrix} - \begin{bmatrix} x_1 \\ -x_2 \ln x_1 \end{bmatrix} = \begin{bmatrix} -x_1 \\ 0 \end{bmatrix}$$

$\text{span}\{g, ad_f g\}\Big|_{x_0} = \text{span}\left\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \end{bmatrix}\right\}$ has dim 2

$\Rightarrow$ cond 1) is satisfied

cond 2) is automatically satisfied since we have a scalar control. $m = 1$.

$\Rightarrow$ system can be linearized by change of state in this case:

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$$z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \Phi(x) = \begin{bmatrix} \ln x_1 \\ \ln x_2 \end{bmatrix}$$

to verify it (and compute A, b)

- $\frac{\partial \Phi(x)}{\partial x} f(x) = A \Phi(x)$ i.e.

$$\begin{bmatrix} \frac{1}{x_1} & 0 \\ 0 & \frac{1}{x_2} \end{bmatrix} \begin{bmatrix} x_1 \ln x_2 \\ -x_2 \ln x_1 \end{bmatrix} = \begin{bmatrix} \ln x_2 \\ -\ln x_1 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} \ln x_1 \\ \ln x_2 \end{bmatrix}}_{\Phi(x)}$$

- $\frac{\partial \Phi}{\partial x} g(x) = b$

$$\begin{bmatrix} 1/x_1 & 0 \\ 0 & 1/x_2 \end{bmatrix} \begin{bmatrix} 0 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = b$$

alternatively: compute $\Phi^{-1}(z) = \begin{bmatrix} e^{z_1} \\ e^{z_2} \end{bmatrix}$

and compute

$$\dot{z} = \frac{\partial \Phi}{\partial x} \dot{x} = \frac{\partial \Phi}{\partial x} \left(f(x) + g(x) u \right) \Big|_{x = \Phi^{-1}(z)}$$

$$= \begin{bmatrix} \frac{1}{x_1} & 0 \\ 0 & \frac{1}{x_2} \end{bmatrix} \left(\begin{bmatrix} x_1 \ln x_2 \\ -x_2 \ln x_1 \end{bmatrix} + \begin{bmatrix} 0 \\ x_2 \end{bmatrix} u \right) \Big|_{x = \begin{bmatrix} e^{z_1} \\ e^{z_2} \end{bmatrix}}$$

resulting linear system is

$$\begin{cases} \dot{z}_1 = z_2 \\ \dot{z}_2 = -z_1 + u \end{cases}$$

//

__Remark__: state linearız (__exact__) $\neq$ Jacobian linearız. (in which you __neglect__ higher order terms)

(117obis)

Actually, in this example exact linearization and Jacobian linearization at x_0 coincide!

$$A_0 = \left.\frac{\partial f}{\partial x}\right|_{x_0 = \begin{bmatrix}1\\1\end{bmatrix}} = \begin{bmatrix} \ln x_2 & \frac{x_1}{x_2} \\ -\frac{x_2}{x_1} & -\ln x_1 \end{bmatrix}_{x_0} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$B_0 = g(x_0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

however if you compute Jacobian lin. its value is only local, while the exact linearization ~~has~~ is valid where $\Phi(0)$ is a diffeomorphism i.e. on the entire $M = \mathbb{R}^2_+$ (open) where the system is defined

Feedback linearization

motivating example:

mechanical system (e.g. robotic arm)

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = u$$

where $M(q) > 0$ matrix of inertia/masses

$C(q,\dot{q})$ = matrix of Coriolis forces

$g(q)$ = gravitational forces

rewrite it as 1st order order ODEs:

$$\frac{d}{dt}\begin{bmatrix} q \\ \dot{q} \end{bmatrix} = \begin{bmatrix} \dot{q} \\ M^{-1}(q)\left(-C(q,\dot{q})\dot{q} - g(q) + u\right) \end{bmatrix}$$

If I choose a change of feedback

$$u = C(q,\dot{q})\dot{q} + g(q) + M(q)v \qquad \text{where } v = \text{new input}$$

$\Rightarrow$ all dynamics are canceled and I get

$$\frac{d}{dt}\begin{bmatrix} q \\ \dot{q} \end{bmatrix} = \begin{bmatrix} \dot{q} \\ v \end{bmatrix} \quad \Rightarrow \text{linear system of } 2n \text{ eq.}$$

$\Rightarrow$ also a change of feedback can linearize the dynamics!

change of feedback is a feedback (i.e. requires to know the state!) It is also called prefeedback or feedforward control

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what kind of nonlinearities can I eliminate in this way?

1) input-multiplicative invertible nonlinearities

$$\dot{x} = Ax + g(x)u \quad \text{with } g(x) = B\theta(x)$$
$$= Ax + \sum_i g_i(x) u_i$$

B $n \times m$ matrix
$\theta(x)$ $m \times m$ matrix invertible

then choosing

$$u = \beta(x) v \quad \text{with } \beta(x) = \theta^{-1}(x)$$

leads to

$$\dot{x} = Ax + Bv$$

2) additive nonlinearities

$$\dot{x} = Ax + B(\gamma(x) + u) \quad \gamma(x) \ n \times 1 \text{ vector}.$$

then $u = -\gamma(x) + v$ leads to

$$\dot{x} = Ax + Bv$$

$\Rightarrow$ change of feedback alone can be used to linearize systems of the form

$$\dot{x} = Ax + B\theta(x)(\gamma(x) + u)$$

by choosing $u(x) = \alpha(x) + \beta(x) v$

with $\alpha(x) = -\gamma(x)$
$\beta(x) = \theta^{-1}(x)$

$$\Rightarrow \dot{x} = Ax + B\cancel{\theta(x)}(\cancel{\gamma(x)} - \cancel{\gamma(x)} + \cancel{\theta^{-1}(x)} v)$$
$$= Ax + Bv$$

Combining change of state $\Phi(x)$ and change of feedback $u = \alpha(x) + \beta(x)v$ we get a full feedback equivalence transf

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Consider
$$\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x) u_i \qquad (*)$$
use
- change of state $z = \Phi(x)$ diffeom.
- change of input $u = \alpha(x) + \beta(x)v$

  $\beta(x)$ invertible

we get

$$\dot{z} = \left(\frac{\partial \Phi}{\partial x} f(x) + \frac{\partial \Phi}{\partial x} \sum g_i(x) u_i\right)\Bigg|_{\substack{x = \Phi^{-1}(z) \\ u_i = \alpha_i(\Phi^{-1}(z)) + \beta_i(\Phi^{-1}(z))v}}$$

(from which $v_i = \dfrac{u_i - \alpha_i(x)}{\beta_i(x)}$)

plug in the input u_i

$$= \frac{\partial \Phi}{\partial x} f(x) + \sum \frac{\partial \Phi}{\partial x} g_i(x) \left(\alpha_i(x) + \beta_i(x) v_i\right)$$

$$= Az + Bv$$

which implies (taking $z = \Phi(x)$ in this last expr.)

$$\begin{cases} \dfrac{\partial \Phi}{\partial x} f(x) + \sum \dfrac{\partial \Phi}{\partial x} g_i(x) \alpha_i(x) = A\Phi(x) \\ \dfrac{\partial \Phi}{\partial x} g_i(x) \beta_i(x) = b_i \end{cases}$$

$\Rightarrow$ System of PDE (difficult to solve!)

Also in this case ∃ geometric character.

From (*) construct the chain of distrib.

$D_1 = \text{span}\{g_1, \dots, g_m\}$

$D_2 = \text{span}\{g_i, ad_f\, g_i,\ i = 1, \dots m\}$

$D_3 = \text{span}\{g_i, ad_f\, g_i, ad_f^2\, g_i,\ i = 1 \dots m\}$

$\vdots$

$D_k = \text{span}\{ad_f^j\, g_i\ \ i = 1, \dots, m,\ j = 0, 1, \dots k-1\}$

this is clearly growing $D_1 \subset D_2 \subset \dots \subset D_k$

<u>thm</u> Consider (*) with $f(x_0) = 0$ and assume local accessibility at x_0 - then (*) is feedback ~~equivalent~~ linearizable around x_0 iff the distributions $D_1, D_2, \dots D_n$ are all involutive and constant dimensional near x_0. The resulting linear system is controllab

<u>Proof (idea)</u> Similar to thm for state linearization, in spirit: the input v.f. g_i and $ad_f^k\, g_i$ have to "behave" like constant v.f.

For the linear system $\dot{z} = A\cancel{z} + Bv$, the dist D_k^{linear} are $D_k^{linear} = Im[B \; AB \ldots A^{k-1}B]$ which are all involutive (because they are const. v. f.).

Since $D_k = \frac{\partial \Phi}{\partial x} D_k^{linear}$ with $\frac{\partial \Phi}{\partial x}$ full rank then so must be also the D_k

When we apply also the feedback $u = \alpha(x) + \beta(x)v$ we pass from f, g_i to

$$\tilde{f}(x) = f(x) + \sum g_i(x)\alpha_i(x)$$
$$\tilde{g}(x) = g_i(x)\beta_i(x)$$

One should show that $D_k(x) = \tilde{D}_k(x)$ for the new distribution (tedious...)

meaning: D_k involutive are feedback invariants (i.e. invariant to the feedback transformations $u = \alpha(x) + \beta(x)v$, but only when D_k are themselves involutive, not in general //

$$D_k \longleftrightarrow \tilde{D}_k \longleftrightarrow \frac{\partial \Phi}{\partial x}\tilde{D}_k = D_k^{linear}$$

example

$$\dot{x}_1 = \gamma \sin x_2$$
$$\dot{x}_2 = -x_1^2 + u \qquad x_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

this cannot be linearized by simple change of feedback, like $u = x_1^2 + v$ which leads to $\begin{cases} \dot{x}_1 = \gamma \sin x_2 \\ \dot{x}_2 = v \end{cases}$

Let us verify that the system is feedback linearizable

$D_1 = \text{span}\{g\} = \text{span}\left\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$ has dim 1 everywhere and is involutive by constr.

$$[f, g] = ad_f g = 0 - \begin{bmatrix} 0 & \gamma\cos x_2 \\ -2x_1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \gamma \cos x_2 \\ 0 \end{bmatrix}$$

$D_2 = \text{span}\{g, ad_f g\} = \text{span}\left\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} \gamma\cos x_2 \\ 0 \end{bmatrix}\right\}$

has dim $= 2$ and const. rank around $x_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\Rightarrow$ D_2 involutive

$\Rightarrow$ we also have LARC cond

$\Rightarrow$ the syst. is feedback linearizable

In fact, we can combine with the state change

$$z = \Phi(x) = \begin{bmatrix} x_1 \\ \gamma \sin x_2 \end{bmatrix} \Rightarrow \frac{\partial \Phi}{\partial x} = \begin{bmatrix} 1 & 0 \\ 0 & \gamma \cos x_2 \end{bmatrix}$$

$$\Rightarrow \dot{z} = \frac{\partial \Phi}{\partial x} \dot{x} = \begin{bmatrix} 1 & 0 \\ 0 & \gamma \cos x_2 \end{bmatrix} \begin{bmatrix} \gamma \sin x_2 \\ -x_1^2 + u \end{bmatrix} \Bigg|_{x = \Phi^{-1}(z)}$$

$$= \begin{bmatrix} \gamma \sin(x_2) \\ \gamma \cos(x_2)\,(-x_1^2 + u) \end{bmatrix} \Bigg|_{x = \Phi^{-1}(z)}$$

If we choose now

$$u = x_1^2 + \frac{v}{\gamma \cos x_2}$$

and use $x = \Phi^{-1}(z) = \begin{bmatrix} z_1 \\ \arcsin\left(\frac{z_2}{\gamma}\right) \end{bmatrix}$ (easy to compute in this case...)

$$\Rightarrow \dot{z} = \begin{bmatrix} \gamma \sin x_2 \\ v \end{bmatrix} \Bigg|_{x = \Phi^{-1}(z)} = \begin{bmatrix} \gamma \sin\left(\arcsin\left(\frac{z_2}{\gamma}\right)\right) \\ v \end{bmatrix} = \begin{bmatrix} z_2 \\ v \end{bmatrix}$$

↑

simpler after change of feedback

$\Rightarrow$ linear system

(Here I do not need $\Phi^{-1}(z)$ – Enough to observe that $z_2 = \gamma \sin x_2$ and plug it into $\dot{z} = \begin{bmatrix} \gamma \sin x_2 \\ v \end{bmatrix}$)

For $\begin{cases} \dot{z}_1 = z_2 \\ \dot{z}_2 = v \end{cases}$ I can now design a stabilizing linear controller $v = Kz = K\Phi(x)$

$\Rightarrow u = \alpha(x) + \beta(x) K \Phi(x) = x_1^2 \; \dfrac{K\Phi(x)}{\gamma \cos x_2}$ is the complete feedback that stabilizes the original system.

Input-Output feedback linearization

motivating ~~example~~: consider again previous example, but with output

$$\begin{cases} \dot{x}_1 = \gamma \sin x_2 \\ \dot{x}_2 = -x_1^2 + u \\ y = x_2 \end{cases}$$

$\leftarrow$ linear output eq.

If we consider ~~only~~ the feedback linearizing ~~change~~ just mentioned

$$z = \Phi(x) = \begin{bmatrix} x_1 \\ \gamma \sin x_2 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \Phi^{-1}(z) = \begin{bmatrix} z_1 \\ \arcsin\left(\frac{z_2}{\gamma}\right) \end{bmatrix}$$

$$u = x_1^2 + \frac{v}{\gamma \cos x_2}$$

then the system + output in the new basis and input is

$$\begin{cases} \dot{z}_1 = z_2 \\ \dot{z}_2 = v \\ y = \arcsin\left(\frac{z_2}{\gamma}\right) \end{cases}$$

$\Rightarrow$ nonlinear output eq.

Sometimes it might be useful to linearize only the input-output dependence not the entire state vector.

this is particularly true when the residual nonlinear part is not influencing the output.

If I just use

$u = x_1^2 + v$ then

$$\begin{cases} \dot{x}_1 = \gamma \sin x_2 \\ \dot{x}_2 = v \\ y = x_2 \end{cases}$$

this part of the dynamics does not affect the output.
input-output part is linear

$\Rightarrow$ the change of input has decoupled x_1 from the output i.e. it has made it <u>unobservable</u> (we did not discuss this notion in class but it corresponds to its linear counterpart).
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Next: we try to build a general theory for input-output feedback linearization, based on these principles.

Assumption : SISO system

$$(*) \begin{cases} \dot{x} = f(x) + g(x)\,u \\ y = h(x) \end{cases}$$

$f, g, h \in C^\infty$

x_0 s.t. $f(x_0) = 0$

y does not depend on u : let us differentiate it until an explicit dependence on u appears.

$$\dot{y} = \frac{\partial h}{\partial x}\dot{x} = \frac{\partial h}{\partial x}\left(f(x) + g(x)u\right)$$

using Lie deriv.

$$= L_f h(x) + L_g h(x)\,u$$

- if $L_g h(x) \neq 0 \quad \forall x \in B(x_0)$

then $u(x) = \frac{1}{L_g h(x)}\left(-L_f h(x) + v\right)$

yields the first order linear system

$$\dot{y} = v$$

$\Rightarrow$ we have achieved input-output linearization : part of the state vector ~~space~~ can be changed to get

$$\begin{cases} \dot{\xi}_1 = v \\ y = \xi_1 \end{cases} \Bigg\} \text{ 1st order integrator}$$

+ rest of the system ..

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- If instead $L_g h(x) = 0 \quad \forall x \in B(x_0)$
 then differentiate again

$$\ddot{y} = \frac{\partial}{\partial x}(L_f h(x))\dot{x}$$
$$= \frac{\partial}{\partial x}(L_f h) f + \frac{\partial}{\partial x}(L_f h) g u$$
$$= L_f^2 h + L_g L_f h\, u$$

- If $L_g L_f h \neq 0$ then choose input

$$u = \frac{1}{L_g L_f h}\left(-L_f^2 h + v\right)$$

$\Rightarrow \ddot{y} = v$ double integrator

$\Rightarrow$ part of the state vector can become

$$\begin{cases} \dot{\xi}_1 = \xi_2 \\ \dot{\xi}_2 = v \\ y = \xi_1 \end{cases}$$

- If $L_g L_f h = 0$ then differentiate again

- Call r = smallest integer for which

$$L_g L_f^{r-1} h(x) \neq 0 \quad \forall x \in B(x_0)$$

~~def The system (*) is said to have relative degree r~~

(n182)

<u>def</u> the system (*) is said to have <u>relative degree</u> r at x_0 if

$$L_g L_f^k h(x) = 0 \quad \forall x \in B(x_0),\ k = 0, 1, \dots r-2$$
$$L_g L_f^{r-1} h(x) \neq 0 \quad \forall x \in B(x_0)$$

the input-output system is then

$$y^{(r)} = L_f^r h(x) + L_g L_f^{r-1} h(x)\, u$$

or written in state space form (r-dimensional)

$$\begin{cases} \dot\xi_1 = \xi_2 \\ \dot\xi_2 = \xi_3 \\ \vdots \\ \dot\xi_r = L_f^r h(x) + L_g L_f^{r-1} h(x)\, u \\ y = \xi_1 \end{cases}$$

Using the feedback $\quad u = \dfrac{1}{L_g L_f^{r-1} h}\left(-L_f^r h(x) + v\right)$

It becomes

$$y^{(r)} = v$$

or, in state space form (r-dimensional)

$$\dot\xi = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ & & \ddots & & \vdots \\ & & & & 0 \\ & & & & 1 \\ 0 & & \cdots & & 0 \end{bmatrix} \xi + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} v$$

$$y = [1\ 0 \cdots 0]\, \xi$$

this is a linear system of rel. degree $= r$ (this is a chain of integrators)

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In SISO linear systems: relative degree =
= excess of poles w.r.t. zeros

From $y^{(r)} = v$
in the Laplace domain, the transfer function is

$$Y(s) = \frac{1}{s^r} U(s)$$

$\Rightarrow$ rel. degree is r -

What is the partial change of state variables that accomplish this?
By construction we can take

$$\phi_1(x) = h(x)$$
$$\phi_2(x) = L_f h(x)$$
$$\vdots$$
$$\phi_r(x) = L_f^{r-1} h(x)$$

(i.e. y and its first r derivatives)
These are "good" candidates for a change of basis

<u>Prop</u> the differentials

$dh(x) = \frac{\partial h}{\partial x}$, $dL_f h(x) = \frac{\partial}{\partial x} L_f h(x)$, ...

.. $dL_f^{r-1} h(x) = \frac{\partial}{\partial x} L_f^{r-1} h(x)$ are linearly independent ("like rows of a Jacobian m.")

To complete the change of basis we must choose other $\phi_{r+1}(x) \dots \phi_n(x)$ so that

$$\begin{bmatrix} \xi \\ - \\ \eta \end{bmatrix} = \Phi(x) = \begin{bmatrix} \phi_1(x) \\ \vdots \\ \phi_r(x) \\ \phi_{r+1}(x) \\ \phi_n(x) \end{bmatrix}$$ is the new state

the Jacobian $\frac{\partial \Phi}{\partial x}$ must be full rank near x_0
$\forall x \in B(x_0)$

One way (choice is not unique) to choose $\phi_{r+1}(x) \dots \phi_n(x)$ is such that the input u does not appear in the ODEs for η

__Prop__ $\exists$ $\phi_{r+1}(x), \dots, \phi_n(x)$ s.t. $\Phi(x)$ is ~~nonsingular~~ a difeomorphism at x_0 and $L_g \phi_i(x) = 0$ $\forall i = r+1, \dots, n$
$\forall x \in B(x_0)$

__meaning__: $\Delta_g(x) = \text{span}\{g(x)\}$ is a regular distribution at $x \in B(x_0)$, of dim 1
$\Rightarrow$ it is involutive $\Rightarrow$ $\exists$ ~~...~~ integral ~~functions~~ manifold of dim 1
$\Rightarrow$ $\exists$ $n-1$ functions $\lambda_1, \dots, \lambda_{n-1}$ s.t.

$\frac{\partial \lambda_i}{\partial x}$ $i = 1, \dots, n-1$ linearly indep.

$$L_g \lambda_i(x) = \frac{\partial \lambda_i}{\partial x} g = 0 \quad i = 1, \dots, n-1$$

⇒ enough to choose for $\phi_i(x)$ any $n-r$ of these functions $\lambda_i(x)$, s.t. $\Phi(x) = \begin{bmatrix} h(x) \\ L_f h(x) \\ L_f^{r-1} h(x) \\ \phi_{r+1}(x) \\ \phi_n(x) \end{bmatrix}$ is diffeom.

Then $\frac{\partial \phi_i(x)}{\partial x} \perp g(x)$ for all $i = r+1, \dots, n$

⇒ when we apply the change of basis the input u does not appear

Infact, if $\eta_i = \phi_i(x)$ $\quad i = r+1, \dots, n$

$$\dot{\eta}_i = \frac{d}{dt} \phi_i(x) = \frac{\partial \phi_i}{\partial x} \dot{x} = \frac{\partial \phi_i}{\partial x} \left(f(x) + g(x) u \right)$$

$$= \frac{\partial \phi_i}{\partial x} f(x) + \underbrace{\frac{\partial \phi_i}{\partial x} g(x)}_{=0 \text{ by construction}} u$$

$$\left(= L_f \phi_i(x) + \underbrace{L_g \phi_i(x)}_{=0} u \right)$$

⇒ in the new basis $\begin{bmatrix} \xi \\ \eta \end{bmatrix} = \Phi(x)$

we have the system

$$\begin{cases} \dot{\xi}_1 = \xi_2 \\ \vdots \\ \dot{\xi}_{r-1} = \xi_r \\ \dot{\xi}_r = L_f^r h(x) + L_g L_f^{r-1} h(x) u = v \\ \dot{\eta} = q(\xi, \eta) \quad \} \text{ n-r dimensional} \end{cases}$$

$$y = \xi_1$$

this is called the <u>normal form</u> of a nonlinear SISO system

Notice how the η part of the state vector does not enter into the output eq.
⇒ the η-subsystem is <u>unobservable</u> from the given output.

example

$$\begin{cases} \dot{x} = \begin{bmatrix} -x_1 \\ x_1 x_2 \\ x_2 \end{bmatrix} + \begin{bmatrix} \exp(x_2) \\ 1 \\ 0 \end{bmatrix} u \qquad x_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\ y = x_3 \end{cases}$$

compute relative degree

$$\dot{y} = \frac{\partial h}{\partial x} \dot{x} = L_f h(x) + L_g h(x) u =$$

$$= \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -x_1 \\ x_1 x_2 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \exp(x_2) \\ 1 \\ 0 \end{bmatrix}}_{=0} u$$

$$= x_2$$

u does not appear

⇒ must differentiate again

$$\ddot{y} = L_f^2 h(x) + L_g L_f h(x) u$$

$$= \frac{\partial (L_f h)}{\partial x} f + \frac{\partial (L_f h)}{\partial x} g u$$

$$= \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} -x_1 \\ x_1 x_2 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \exp(x_2) \\ 1 \\ 0 \end{bmatrix} u$$

$$= x_1 x_2 + u$$ ⇒ input appears

⇒ relative degree = 2

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Change of basis: first part (known)

$$\xi_1 = \phi_1(x) = h(x) = x_3$$

$$\xi_2 = \phi_2(x) = L_f h(x) = x_2$$

while for the second part $\eta = \phi_3(x)$ we must find function $\phi_3(x)$ s.t. $L_g \phi_3(x) = 0$

$$L_g \phi_3 = \frac{\partial \phi_3}{\partial x} g(x) = \frac{\partial \phi_3}{\partial x} \begin{bmatrix} \exp(x_2) \\ 1 \\ 0 \end{bmatrix} =$$

$$= \left(\frac{\partial \phi_3}{\partial x_1} \quad \frac{\partial \phi_3}{\partial x_2} \quad \frac{\partial \phi_3}{\partial x_3} \right) \begin{bmatrix} \exp(x_2) \\ 1 \\ 0 \end{bmatrix} = \frac{\partial \phi_3}{\partial x_1} \exp(x_2) + \frac{\partial \phi_3}{\partial x_2} = 0$$

$\Rightarrow$ must solve a PDE (may be difficult in general...)

In this case it is easy:

$$\phi_3(x) = 1 + x_1 - \exp(x_2)$$

In fact

$$L_g \phi_3 = \frac{\partial \phi_3}{\partial x} g = \begin{bmatrix} 1 & -\exp(x_2) & 0 \end{bmatrix} \begin{bmatrix} \exp(x_2) \\ 1 \\ 0 \end{bmatrix} = 0$$

the whole change of basis is then

$$\begin{bmatrix} \xi_1 \\ \xi_2 \\ \eta \end{bmatrix} = \Phi(x) = \begin{bmatrix} \phi_1(x) \\ \phi_2(x) \\ \phi_3(x) \end{bmatrix} = \begin{bmatrix} x_3 \\ x_2 \\ 1 + x_1 - \exp(x_2) \end{bmatrix}$$

Its Jacobian $\dfrac{\partial \Phi}{\partial x} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & -\exp(x_2) & 0 \end{bmatrix}$ has full rank $\forall x$

$\Rightarrow$ global transf.

Its inverse $\Phi^{-1}(z)$ is also easy to compute

$$\begin{cases} x_1 = -1 + \eta + \exp(\xi_2) \\ x_2 = \xi_2 \\ x_3 = \xi_1 \end{cases}$$

System in normal form is

$$\begin{cases} \dot{\xi}_1 = \xi_2 \\ \dot{\xi}_2 = L_f^2 h(x) + L_g L_f h(x) u = x_1 x_2 + u \\ \dot{\eta} = L_f \phi_3(x) = \frac{\partial \phi_3}{\partial x} f(x) = \begin{bmatrix} 1 & -\exp(x_2) & 0 \end{bmatrix} \begin{bmatrix} -x_1 \\ x_1 x_2 \\ x_2 \end{bmatrix} \\ \quad = -x_1 - \exp(x_2) x_1 x_2 \end{cases}$$

on the r.h.s. I must now replace x with (ξ, η)

$$\begin{cases} \dot{\xi}_1 = \xi_2 \\ \dot{\xi}_2 = \underline{(-1 + \eta + \exp \xi_2)\, \xi_2 + u} = v \end{cases}$$

this I can skip computing if I apply first the change of input!

$$\dot{\eta} = +1 - \eta - \exp(\xi_2) - \exp(\xi_2)(-1 + \eta + \exp(\xi_2))\, \xi_2$$

$$= \big(1 - \eta - \exp(\xi_2)\big)\big(1 + \xi_2 \exp(\xi_2)\big)$$