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PRINCIPAL ANGLES IN SYSTEM THEORY, INFORMATION THEORY AND SIGNAL PROCESSING

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het behalen van het doctoraat
in de toegepaste wetenschappen

door

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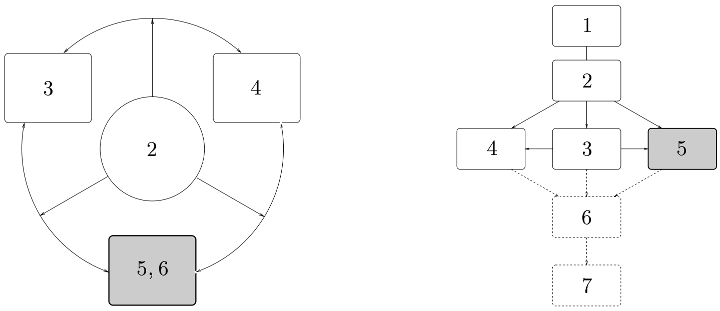
Chapter 5

The Cepstrum



There is only one missing link left in order to close the circle in the left picture above, which is the signal processing part, spread over Chapter 5 and Chapter 6. Indeed, in order to define the cepstral norm in the next chapter, we first need to define the cepstrum and to explain its properties. From Figure 1.10 and the right picture above, we see that this chapter forms a bit an excursion from the main path. As a matter of fact, this is the only chapter where we do not encounter principal angles between subspaces nor canonical correlations. However, it will become clear that we will need the cepstrum and its properties for the derivations in Chapter 6.

Contributions We try to give a clear and concise overview of the power cepstrum and the complex cepstrum, interlaced with examples to illustrate their applicability. Furthermore, we point out how the cepstrum can be used for the identification of linear models. There exists a simple relation between the Markov parameters of the model and its cepstrum. Consequently, one could obtain an estimate of the Markov parameters by first estimating a number of cepstrum coefficients from measurements. A state space description of the identified model can then be obtained by applying an algorithm for the deterministic realization problem (see Section 3.3.1).

Chapter overview We start with a historical introduction in Section 5.1. In Section 5.2, we define the power cepstrum. The complex cepstrum is treated in Section 5.3. The power cepstrum of an ARMA process can easily be expressed in terms of its poles and zeros. This is done in Section 5.4. In Section 5.5 we discuss the inverse problem, where an ARMA model is estimated from a series of cepstral coefficients. This is an identification problem, based on the cepstrum.

5.1 Introduction

In 1963, Bogert, Healy and Tukey published a paper with the unusual title 'The quefrency alanysis of time series for echoes: cepstrum, pseudo-autocovariance, cross-cepstrum, and saphe cracking', in which they introduced the cepstrum and showed that it is a useful tool for the detection of echoes in seismological data [17]. As can be seen from the title, they proposed a whole new terminology by paraphrasing existing words. The terms that are still in use nowadays, are given in Table 5.1. We will only adopt the term 'cepstrum' for the inverse Fourier transform of the logarithm of the spectral density[1].

spectrum	cepstrum
filtering	liftering
frequency	quefrency
harmonics	rahmonics

Table 5.1: A few signal processing terms and their paraphrases as introduced in [17].

Meanwhile, Oppenheim had been developing his theory of homomorphic[2] systems [110], for which soon applications in signal processing were found, such as image enhancement [139], speech analysis [112] and blind deconvolution. A classic example of blind deconvolution is the restoration of the old phonograph recordings of Caruso [140], where the horn distortions are removed from the

[1] More precisely, we will call this the 'power cepstrum', in order to make a clear distinction with the complex cepstrum (see further).

[2] A homomorphism is a mapping that preserves all relevant structure. If A and B are groups with respective operations $\bigcirc$ and $\Box$, then f is a homomorphism of A into B if $f(a_1 \bigcirc a_2) = f(a_1) \Box f(a_2)$ for all $a_1, a_2 \in A$. We are only interested in the homomorphic system for the convolution of signals. By converting two convolved signals to the cepstral domain, the converted signals are added and can be treated with the usual filter techniques designed for added signals.

audio signal[3].
The cepstrum analysis as described by Bogert et al. turned out to be a special case of Oppenheim's homomorphic signal processing. Oppenheim takes the complex logarithm of the Z-transform of a signal, whereas Bogert et al. consider the logarithm of the power spectrum, which is a real function. Therefore, Oppenheim's version of the cepstrum is called the 'complex cepstrum', although the cepstrum itself is a real function (see Section 5.3).
An interesting document to get to know the circumstances and spirit of the times in which these concepts were developed, is the interview with Oppenheim, conducted for the IEEE History Center [54].
A good introduction to homomorphic signal processing is Chapter 10 in the book by Oppenheim and Schafer [113]. They start with an abstract description of homomorphic systems, consider systems for the multiplication and convolution of signals, discuss the complex cepstrum and end up with a number of interesting applications. Deller et al. [41, Chapter 6] treat cepstral analysis for speech signals, emphasizing the signal processing aspects, which we will not treat in detail.
Besides the above mentioned applications, the cepstrum is also used for machine diagnostics because of its ability to detect periodicities in the spectrum [161] and for measuring the distance between two signals, see [6] and references therein and [105]. The application of the cepstrum to distance measures will be treated in Chapter 6.
Oppenheim [111] already indicated that the cepstrum can also be used for the identification of a stochastic model (see Section 3.3 for an introduction to stochastic system identification). We will describe his approach in Section 5.5. Recently, Byrnes et al. [23, 22] and Enqvist [46, 47] showed that the cepstrum coefficients can be applied to find solutions for the rational covariance extension problem [23, 22, 46, 47].

5.2 The power cepstrum

As we have mentioned in the introduction of this chapter, the power cepstrum originated in the work of Bogert et al. [17] as a solution for the detection of echoes in seismic data. Before we give the formal definitions of the power cepstrum, we consider an example where it is introduced naturally and the need for two definitions becomes clear. We construct in Example 5.1 a signal with one echo and show how the cepstrum reveals this echo, whereas it is difficult to detect it in the autocovariance function. Next, we give two definitions for the power cepstrum, the first one defines the power cepstrum of a stochastic

[3]The result of this restoration can be heard via the video of the Research Laboratory of Electronics of the MIT, available on `http://rleweb.mit.edu/video/video.htm`. The audio fragment is contained in video clip 10 'Digital Signal Processing – continued'.

process and the second one the power cepstrum of an absolutely summable sequence.

Example 5.1. Echo detection
The detection (and cancellation) of echoes is an important problem in many areas. The original application in [17] dealt with seismic data. But also in most communication channels (telephony, wireless communication, ...), echoes appear as the result of e.g. an impedance mismatch along the telephone line, or due to leakage from the loudspeaker to the microphone in e.g. hands-free telephony. State-of-the-art signal processing techniques for echo cancelling in modern ADSL modems are described in [148, 149] and references herein. Acoustic echo cancelling methods can e.g. be found in [44, 45, 126] and references therein.
A signal with one echo can be modeled as $z(k) = y(k) + \gamma y(k - D)$, where γ is the attenuation coefficient ($|\gamma| < 1$) and D is the delay. The purpose of echo detection is to determine D, the time at which the echo appears. The signal $z(k)$ can also be viewed as the convolution of $y(k)$ and the signal $d(k)$ which consists of two pulses:

$$d(k) = \begin{cases} 1 & k = 0 \ , \\ \gamma & k = D \ , \\ 0 & \text{otherwise} \ . \end{cases} \tag{5.1}$$

Thus,

$$z(k) = d(k) * y(k) \ .$$

In this example, the time series $\{y(k)\}$ is obtained by sending white noise with variance 1 through a fourth order ARMA model with transfer function $H(z)$, see Figure 5.1. To make it concrete, the model can be viewed as the model of a transmission channel and the time series $\{y(k)\}$ is the signal that is sent. The

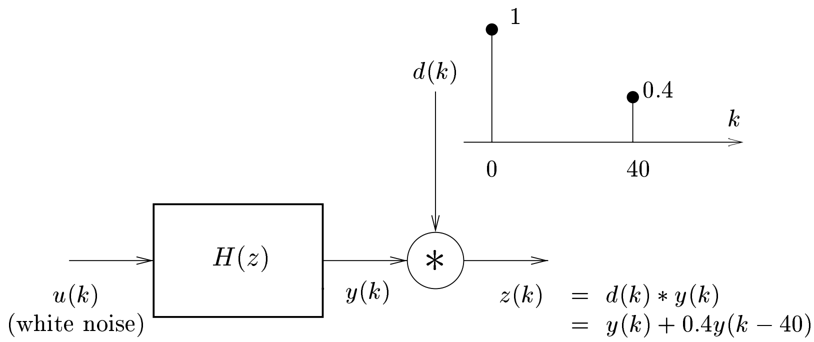


Figure 5.1: The construction of the signal $z(k)$, which is equal to $y(k)$ with an echo of $y(k)$ at $k = 40$. The attenuation coefficient is 0.4. It can be viewed as the convolution of $y(k)$ and $d(k)$ which consists of two pulses, one at $k = 0$ with magnitude 1 and the other at $k = 40$, valued 0.4.

poles of the ARMA(4,4) model are 0.9095, −0.9289 and $-0.4193 \pm 0.8525\,i$, and the zeros are $-0.6668 \pm 0.2835\,i$ and $0.6192 \pm 0.1494\,i$. The attenuation coefficient is chosen to be $\gamma = 0.4$ and the time at which the echo appears is $D = 40$.

The discrete time signals we work with, are shown in the top row of Figure 5.3. As will become clear, the power cepstrum is the obvious tool for detecting the echo in $\{z(k)\}$. The different steps that are taken to compute the power cepstrum of $\{z(k)\}$ are given in Figure 5.2. We discuss each step in detail.

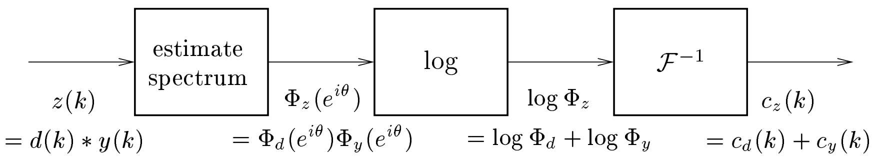


Figure 5.2: The computation of the power cepstrum of $\{z(k)\}$.

1. First, the power spectrum of $\{z(k)\}$, denoted by $\Phi_z(e^{i\theta})$, is estimated. In this example, we will use the theoretical values. Since we know the relation of $\{z(k)\}$ and $\{y(k)\}$, we obtain, via the autocovariance function $R_z(\tau)$, the exact expression for $\Phi_z(e^{i\theta})$ as a function of the power spectrum of $\{y(k)\}$, denoted by $\Phi_y(e^{i\theta})$:

$$\begin{aligned}\Phi_z(e^{i\theta}) &= \mathcal{F}(R_z) &(5.2)\\ &= \mathcal{F}\left((1+\gamma^2)R_y(\tau) + \gamma R_y(\tau + D) + \gamma R_y(\tau - D)\right)\\ &= \left(1 + \gamma^2 + 2\gamma\cos(D\theta)\right)\Phi_y(e^{i\theta})\ . &(5.3)\end{aligned}$$

We now note that $d(k)$ can be viewed as the impulse response of the MA model with transfer function $\frac{z^D+\gamma}{z^D}$. The squared magnitude of its Fourier transform is therefore equal to the power spectral density of the model's output when driven by white noise with variance 1 (see Example 3.2). This output process of the MA model is denoted by $\{y_d(k)\}$ and its spectral density by $\Phi_d(e^{i\theta})$, although it is not the spectral density of $d(k)$

$$\begin{aligned}\Phi_d(e^{i\theta}) &= \mathcal{F}(R_{y_d}) &(5.4)\\ &= |\mathcal{F}(d)|^2\ ,\\ &= 1 + \gamma^2 + 2\gamma\cos(D\theta)\ . &(5.5)\end{aligned}$$

Using (5.5) in (5.3) gives

$$\Phi_z(e^{i\theta}) = \Phi_d(e^{i\theta})\Phi_y(e^{i\theta})\ .$$

The three spectral densities for this example, Φ_z, Φ_d and Φ_y, are shown in the second row of pictures of Figure 5.3. In the first picture we see the

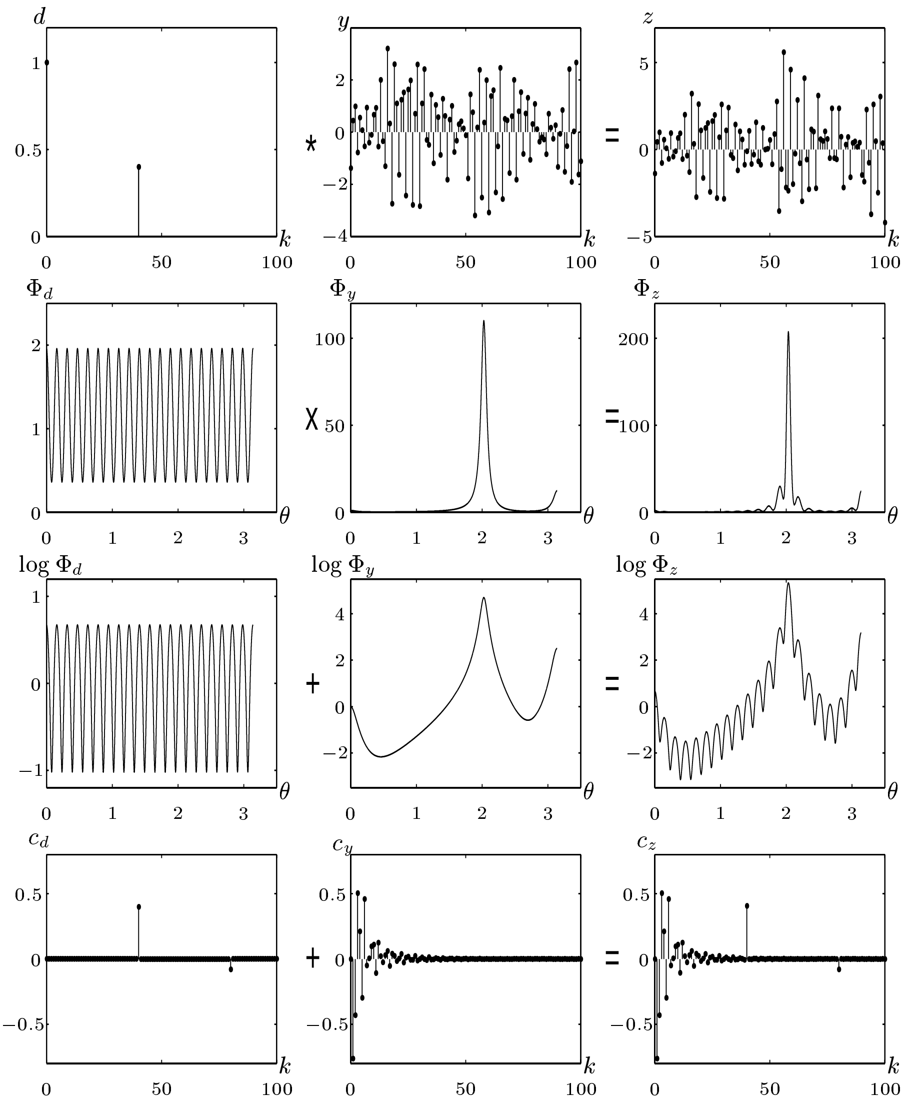


Figure 5.3: The different signals we work with in Example 5.1. In the third column, the different steps to obtain the power cepstrum of $\{z(k)\}$ are recognized. We start from the time signal z, then compute the power spectral density $\Phi_z(e^{i\theta})$, we take the logarithm and transform back to the time domain to obtain $c_z(k)$. In the horizontal direction we see how each z-signal is composed of the corresponding d- and y-signal. In the time domain $d(k)$ and $y(k)$ are convolved, the spectra Φ_d and Φ_y are multiplied, the log spectra and the cepstra are added.

cosine function with period $\frac{2\pi}{40} \approx 0.1571$. The second picture gives the spectral density of the ARMA process $\{y(k)\}$ with a resonance peak at $\theta \approx 2.0279$, i.e. the angle of the complex conjugated pair of poles. The spectral density of $\{z(k)\}$ is the product of these two functions, as shown in the third picture.
In practice, only Φ_z (and not Φ_d nor Φ_y) is known. The influence of the echo (γ and D) is not easily determined from Φ_z.

2. The second step in Figure 5.2 consists of taking the logarithm[4] of the power spectrum of $\{z(k)\}$, by which we obtain an additive combination:

$$\log \Phi_z(e^{i\theta}) = \log \Phi_d(e^{i\theta}) + \log \Phi_y(e^{i\theta}) \ .$$

The first term is equal to

$$\log \Phi_d(e^{i\theta}) = \log(1 + \gamma^2 + 2\gamma \cos(D\theta)) \ ,$$

which is still a periodic function with period $\frac{2\pi}{D}$, see the first picture on row 3 of Figure 5.3. It causes a periodic ripple in $\Phi_z(e^{i\theta})$, which can be seen in the third picture of row 3 in Figure 5.3. This ripple is more clearly seen than the one in the power spectrum (last picture on row 2). The parameter D could now be estimated by determining the period of $\log \Phi_z(e^{i\theta})$.
In practice, however, the ripple will be obscured by noise, stemming from the spectrum estimation and the measurements of $\{z(k)\}$. One of the techniques to determine periodic phenomena in a noisy signal is to look at the Fourier transform of the signal. Since we are already in the frequency domain, we can as well transform $\log \Phi_z(e^{i\theta})$ back to the time domain, which is the last step in Figure 5.2.

3. By taking the inverse Fourier transform of $\log \Phi_z$, we obtain a discrete time signal with spikes at $D, 2D, 3D, \ldots$, which can be seen in the third picture on the last row of Figure 5.3. The explanation is simple. Since $\log \Phi_z$ is equal to the sum of the logarithms of the spectra of $\{y_d(k)\}$ and $\{y(k)\}$, we have

$$\mathcal{F}^{-1}(\log \Phi_z) = \mathcal{F}^{-1}(\log \Phi_d) + \mathcal{F}^{-1}(\log \Phi_y) \ .$$

The inverse Fourier transform of the logarithm of the power spectrum is called the power cepstrum. Denoting the power cepstra of $d(k)$, $\{y(k)\}$ and $\{z(k)\}$, respectively by $c_d(k)$, $c_y(k)$ and $c_z(k)$, gives

$$c_z(k) = c_d(k) + c_y(k) \qquad k \in \mathbb{Z}.$$

Recall that $\Phi_d(e^{i\theta})$ is not the power spectrum of $d(k)$, but of the MA process $\{y_d(k)\}$, obtained by sending white noise with variance 1 through

[4] In this chapter we use the logarithm with base e.

the MA filter with impulse response $d(k)$. Consequently, $c_d(k)$ is strictly spoken not the power cepstrum of $d(k)$, but of the related MA process. It is easy to show (a more general assertion is proven in Proposition 5.1) that $c_d(k)$ is a sequence of pulses at multiples of D (the rahmonics):

$$c_d(k) = \begin{cases} (-1)^{|n|+1}\dfrac{\gamma^{|n|}}{|n|} & k = nD \quad (n \in \mathbb{Z}), \\ 0 & \text{otherwise,} \end{cases}$$

which is shown in the first picture of the last row in Figure 5.3. The sequence of pulses are also observed in the power cepstrum of $\{z(k)\}$, shown in the last picture of Figure 5.3, because the cepstrum of $\{y(k)\}$ dies out sufficiently fast. It will be harder to see the pulses when the decay of $c_y(k)$ is slower. In this example, we can estimate the echo parameter D from the power cepstrum of $\{z(k)\}$. Moreover, we get an indication of the attenuation coefficient's value by measuring the magnitude of the spikes.

The autocovariance function of $\{z(k)\}$ is equal to the inverse Fourier transform of its power spectral density (see (5.2)):

$$R_z = \mathcal{F}^{-1}(\Phi_z) \ .$$

Thus, the cepstrum differs from the autocovariance only by the logarithmic conversion of the spectrum (see also Table 5.2). The autocovariance by itself can also be used for echo detection. Since

$$R_z(\tau) = (1+\gamma^2)R_y(\tau) + \gamma R_y(\tau + D) + \gamma R_y(\tau - D) \ ,$$

we get in R_z an echo of R_y at $\tau = D$, which can be detected if $R_y(\tau)$ decreases fast enough with respect to D, or, if D is large enough with respect to the time

autocovariance		power spectrum
$R(\tau)$	$\overset{\mathcal{F}}{\underset{\mathcal{F}^{-1}}{\rightleftarrows}}$	$\Phi(e^{i\theta})$
one to one $\Updownarrow$		exp $\uparrow\downarrow$ log
$c(k)$	$\overset{\mathcal{F}^{-1}}{\underset{\mathcal{F}}{\leftrightarrows}}$	$\log\Phi(e^{i\theta})$
power cepstrum		log spectrum

Table 5.2: Relations of autocovariance, power spectrum and power cepstrum.

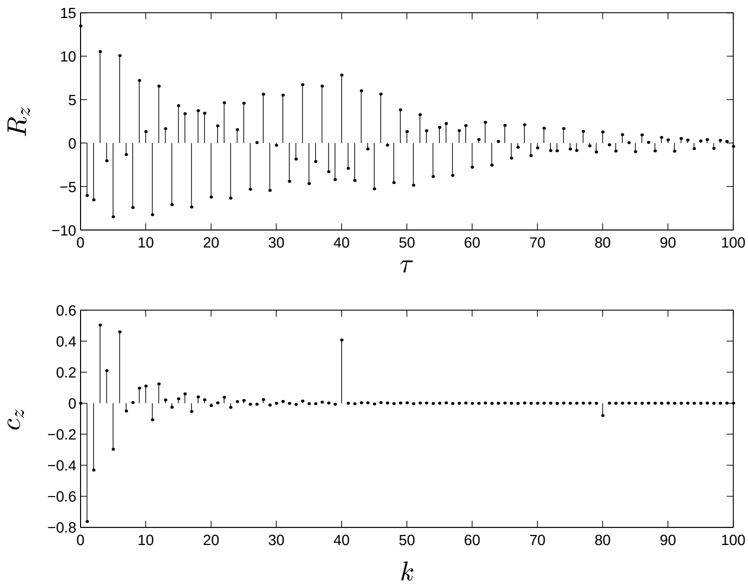


Figure 5.4: Comparison of the autocovariance function $R_z(\tau)$ and the power cepstrum $c_z(k)$ of the stochastic process $\{z(k)\}$. The echo is more clearly visible in the power cepstrum than in the autocovariance. This is partly due to the faster decay of the cepstrum.

constant of R_y. In our example, however, the influence of R_y is still too large at $\tau = D$ to detect the echo clearly. In Figure 5.4 we compare the autocovariance function R_z (top picture) and the power cepstrum c_z (bottom) of $\{z(k)\}$. It is not easy to observe the echo in the autocovariance, but, as we have seen in Figure 5.3, there is a clear peak at $k = 40$ in the cepstrum. Moreover, the 'rahmonic' at $k = 80$ is also an indication of the echo at $k = 40$.
The difference of autocovariance and cepstrum can be understood as follows: the autocovariance is dominated by the large peak in the spectrum Φ_z, while in $\log \Phi_z$ the resonance peak is attenuated and more details of the echo show up. From Figure 5.4 we also see that the cepstrum of $\{y(k)\}$ decays faster than its autocovariance (see also Section 5.4).
Finally, we remark that in more difficult echo detection problems, filtering the log spectrum signal by an appropriate filter (i.e. applying a lifter) can improve the result considerably [17]. $\Diamond$

We can now formally give the definition of the power cepstrum.

Definition 5.1. The power cepstrum of a stationary stochastic process
The power cepstrum of a stationary scalar stochastic process $\{y(k)\}$ with spectral density Φ_y is defined as the inverse Fourier transform of the logarithm of

its power spectral density:

$$c_y(k) = \frac{1}{2\pi}\int_0^{2\pi} \log(\Phi_y(e^{i\theta}))e^{ik\theta}\, d\theta \ .$$

This can also be written as

$$\log \Phi_y(e^{i\theta}) = \sum_{k=-\infty}^{\infty} c_y(k)e^{-ik\theta} \ ,$$

and hence the kth cepstral coefficient is the coordinate of $\log \Phi_y$ w.r.t. the basis signal $e^{-ik\theta}$.

Recall that the spectral density is the Fourier transform of the autocovariance function. For the stationary stochastic process $\{y(k)\}$ with autocovariance $R_y(\tau)$, an alternative expression for the power cepstrum is therefore

$$c_y = \mathcal{F}^{-1}\left(\log \mathcal{F}(R_y)\right) \ .$$

Note that we start from a signal in the time domain (R_y), transform it to the frequency domain to obtain the spectrum ($\Phi_y = \mathcal{F}(R_y)$), and after taking the logarithm, we transform it back to the time domain and obtain the cepstrum ($c_y = \mathcal{F}^{-1}(\log \Phi_y)$). As already mentioned in Example 5.1, the logarithm is the essential difference between the autocovariance and the cepstrum. From a homomorphic processing point of view (see footnote 2 on page 188) the logarithm is necessary to transform the convolved signals to cepstra that are added. In some applications, however, taking the logarithm of the spectrum can also be motivated more directly. For instance, decibels are used when measuring the loudness of sounds because our sense of hearing is roughly logarithmic. Another example is the Richter scale, which measures the energy released by an earth quake [124] (see also [136]). In controller design, but also in modal analysis, one most often displays the Bode plot, which shows the logarithm of the transfer function's magnitude and its phase as a function of the frequency. For stochastic processes this boils down to the logarithm of the spectrum, as was shown in Figure 3.4, where the logarithmic spectrum of the transmitter mast's data is given.

As we have noticed in Example 5.1 with $d(k)$, it is desirable to have a definition for the power cepstrum of sequences that have a Fourier transform themselves and for which one should not rely on the autocovariance. For a sequence $s(k)$ that is absolutely summable, the sum $\sum_{k=-\infty}^{\infty} s(k)e^{-ik\theta}$ converges and hence its Fourier transform exists. An example of an absolutely summable sequence is the impulse response of a linear stable model, of which the Fourier transform is the model's transfer function evaluated on the unit circle. The power cepstrum of absolutely summable sequences can be defined as follows.

Definition 5.2. The power cepstrum of an absolutely summable sequence

The power cepstrum of an absolutely summable sequence $s(k)$ whose Fourier *transform* $S(e^{i\theta})$ *satisfies* $S(e^{i\theta}) \neq 0 \; \forall\theta$, *is defined as the inverse Fourier transform of the logarithm of the magnitude squared of its Fourier transform:*

$$c_s(k) = \frac{1}{2\pi} \int_0^{2\pi} \log(|S(e^{i\theta})|^2) e^{ik\theta} \, d\theta \ ,$$

where $S(e^{i\theta}) = \sum_{k=-\infty}^{\infty} s(k) e^{-ik\theta}$.

Remark 5.1. We take the magnitude squared of $S(e^{i\theta})$ and not just its magnitude, as e.g. in [41], because we want Definition 5.2 to be consistent with Definition 5.1. In particular, if $s(k)$ is the impulse response of an ARMA model, then with Definitions 5.1 and 5.2 its power cepstrum is equal to the power cepstrum of the ARMA process that is the output of the model when it is driven by white noise with variance 1. ○

The power cepstrum is a real and even function:

$$c(k) = c(-k) \in \mathbb{R} \ . \tag{5.6}$$

The estimation of the power cepstrum of a stationary stochastic process

Parametric as well as non-parametric power spectral density estimates can be used for the estimation of the power cepstrum of a stationary stochastic process (see Remark 3.5). The most commonly used parametric power spectral density estimate is obtained from AR modeling of the signal. Once a model is identified, the cepstrum can be computed from the model parameters directly, as we will see in Section 5.4.
Examples of non-parametric power spectral density estimates used in cepstral representation are the periodogram and the smoothed periodogram. We now describe how the power cepstrum of a stationary process can be estimated from a finite sequence of observations via the periodogram (without smoothing). For a given sequence $y(0), \ldots, y(K-1)$ with discrete Fourier transform (DFT)

$$Y(n) = \sum_{k=0}^{K-1} y(k) e^{-i\frac{2\pi}{N}nk} \quad (n = 0, \ldots, K-1) \ ,$$

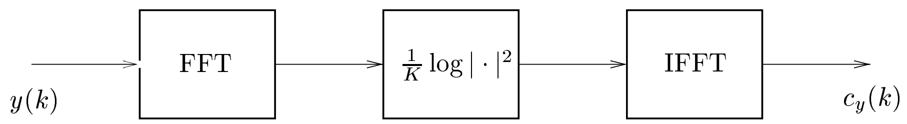


Figure 5.5: Estimation of the power cepstrum.

the periodogram estimate of the spectrum is $S_y(n) = \frac{1}{K}|Y(n)|^2$, and the cepstral components are obtained as the inverse DFT of the logarithm of the periodogram:

$$c_y(k) = \frac{1}{K}\sum_{n=0}^{K-1} \log\left(\frac{1}{K}|Y(n)|^2\right) e^{i\frac{2\pi}{N}nk} \quad (k = 0, \ldots, K-1) .$$

By using the fast Fourier transform algorithm (FFT) [28], we obtain a computationally efficient method, depicted in Figure 5.5.

Example 5.2. The cepstrum of the mast data
As an example, we compute the power cepstrum of the data used in Example 3.1, which are acceleration measurements on a steel transmitter mast. In Figure 5.6 a non-parametric estimate is compared to a parametric estimate. The non-parametric estimate is based on the periodogram of the 3839 measurements and it is obtained by the method in Figure 5.5. For the parametric estimate, we use the 12th order ARMA model, identified in Example 3.1 and of which the autocovariance function and the spectrum are shown in Figure 3.4. From the model, the power cepstrum is obtained by applying the formulas in Proposition 5.1 below. ◊

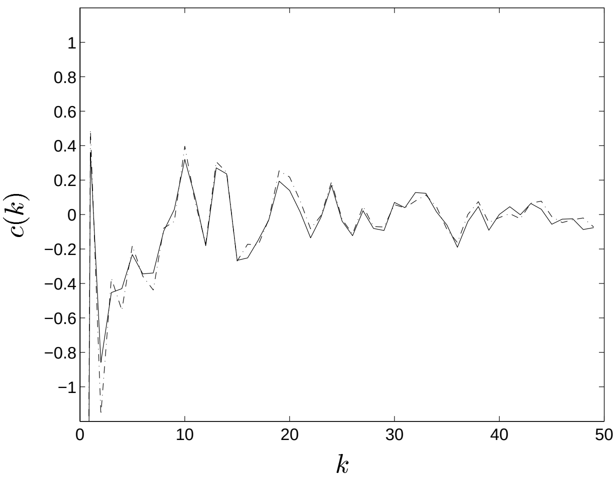


Figure 5.6: A non-parametric (full line) and a parametric (dash-dot line) estimation of the power cepstrum of the mast measurements, introduced in Example 3.1. The model used for the parametric estimation is the 12th order ARMA model that we identified in Example 3.1.

5.3 The complex cepstrum

Sometimes, it is useful not only to detect the echoes, but also to remove them and recover the original signal without echoes. Or, in general, to deconvolve two signals. Using the power cepstrum, however, one discards the phase information and an inversion of the operations is thus not possible. The complex cepstrum does not have this disadvantage. Again, we start with a motivational example and give the definition of the complex cepstrum afterwards.

Example 5.3. Echo detection and removal
Let the signal $s(k)$ be equal to $h(k) + 0.4h(k-40)$, where $h(k)$ is the impulse response of the ARMA model with poles and zeros as in Example 5.1. So, instead of sending white noise through the ARMA model of Example 5.1, we consider its impulse response. The signal $s(k)$ can be viewed as the convolution of $d(k)$, defined in (5.1), and $h(k)$. It contains an echo of $h(k)$ at $k = 40$ with intensity 0.4. Assume that we want to restore the original signal without echo, i.e. $h(k)$.
We know how to detect the echo in the power cepstrum $c_s(k)$ and by filtering the cepstrum with a comb filter, which boils down to leaving out the rahmonics produced by the echo, it can be removed from the cepstrum. Transforming the adapted cepstrum back to the time domain is, however, not possible because the magnitude of the Fourier transform of $s(k)$ was taken to obtain $c_s(k)$ (see Definition 5.2) and the phase information has been lost. Yet, instead of taking the squared magnitude, one could as well work with the Fourier transform $S(e^{i\theta})$ (or the more general Z-transform) directly, which then results in the complex cepstrum $\bar{c}_s(k)$.
A diagram of the procedure to obtain the complex cepstrum of $s(k)$ is given in Figure 5.7. Note that the logarithm of the complex function $S(z)$ is taken. Hence its name 'complex' cepstrum. The complex logarithm should be defined properly, as will be discussed further.

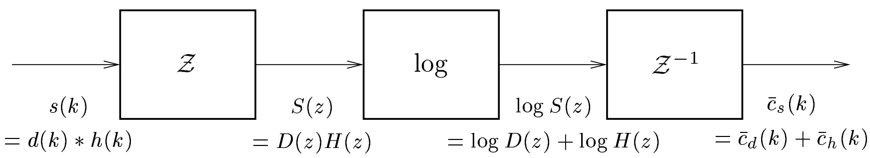


Figure 5.7: Schematic presentation of the computation of the complex cepstrum of $s(k)$.

As will be shown in Corollary 5.2, the complex cepstra of $d(k)$ and $h(k)$ are equal to the power cepstra of $d(k)$, respectively $\{y(k)\}$ in Example 5.1 for $k \geq 0$. Consequently, $\bar{c}_s(k)$ is equal to $c_z(k)$ for $k \geq 0$, which can be seen in Figure 5.4. After removing the spikes at multiples of $D = 40$, we transform the adapted complex cepstrum, denoted by $\bar{c}_s^*(k)$ back to the time domain as in Figure 5.8.

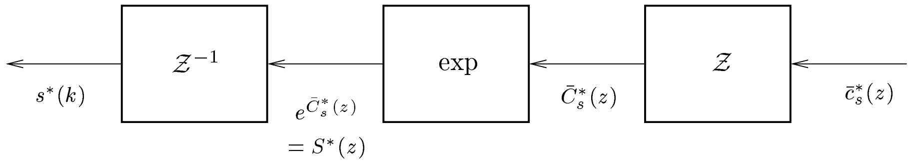


Figure 5.8: Schematic presentation of the inverse operation: going from the cepstral domain back to the time domain.

The result of the echo-removal can be seen in Figure 5.9, where $h(k)$, $s(k)$ and $s^*(k)$ are shown in the top picture and the difference between $s^*(k)$ and $h(k)$ in the bottom picture. We managed to deprive $s(k)$ of its echo pretty well. $\Diamond$

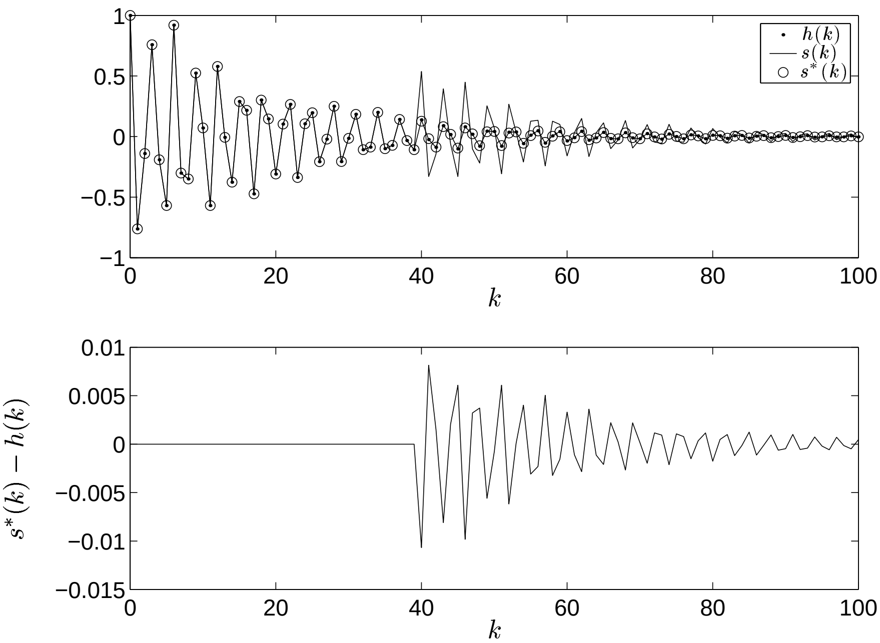


Figure 5.9: The result of the echo-removal from $s(k)$. In the top picture we show three signals: the signal without echo (h), represented by the dots, the signal with an echo at $k = 40$ (s) in full line and the signal from which the echo is removed (s^*) by the circles. The second picture gives the difference of s^* and h, i.e. the error of the restored signal compared to the original echo-free signal. Note that the scales in the two pictures is different!

The example was given as a motivation for the following definition of the complex cepstrum (see also Figure 5.7).

Definition 5.3. The complex cepstrum
Let $s(k)$ be an absolutely summable sequence with Z-transform $S(z)$. The complex cepstrum $\bar{c}_s(k)$ of $s(k)$ is defined as the inverse Z-transform of the complex logarithm of $S(z)$:

$$\bar{c}_s(k) = \frac{1}{2\pi i}\oint_C \log(S(z))z^{k-1}\,dz \ ,$$

where the complex logarithm of $S(z)$ is appropriately defined and the contour C is the unit circle.

Remark 5.2. Since a stationary stochastic process $\{y(k)\}$ does not have a Z-transform, its complex cepstrum cannot be defined. The complex cepstrum of the autocovariance function $R_y(\tau)$ of the process does exist and it is equal to the power cepstrum of $\{y(k)\}$ because the Fourier transform of $R_y(\tau)$ is the power spectrum of $\{y(k)\}$

$$\bar{c}_{R_y}(k) = c_y(k) \ .$$

○

Remark 5.3. The complex logarithm
Let z be a complex number with magnitude r and phase φ, thus $z = re^{i\varphi}$. The logarithm of z is then equal to

$$\log(re^{i\varphi}) = \log r + i(\varphi + k2\pi) \qquad k \in \mathbb{Z} \ .$$

While there is no ambiguity in defining the real part of the complex logarithm ($\log r$), the imaginary part is multi-valued.
In defining the complex cepstrum of a sequence $s(k)$, the complex logarithm of its Z-transform $S(z)$ is taken, which gives

$$\log S(z) = \log|S(z)| + i\log\left(\arg(S(z)) + k2\pi\right) \ ,$$

where $\arg(S(z))$ is the phase of $S(z)$. Hence, a problem of uniqueness arises in the definition of the complex cepstrum of a sequence $s(k)$ because any multiple of 2π can be added to any point of $\arg(S(z))$ without changing the original signal $s(k)$, but the complex cepstrum is altered by such phase changes.
The problem is overcome by demanding that on the unit circle the phase of $S(z)$ be an odd, continuous function of θ with period[5] 2π (for details see [113, pp. 495–497]). This implies that the complex cepstrum $\bar{c}_s(k)$ is a real (not necessarily even) function. The procedure ensuring that $\arg(S(e^{i\theta}))$ has the demanded properties is called 'phase unwrapping', for which algorithms are discussed in [147]. The inverse transformation from cepstral domain to time domain as illustrated in Figure 5.8, does not cause any problems since the (complex) exponential function is single-valued. ○

[5] The period of $\arg(S(e^{i\theta}))$ can also be smaller than 2π, as long as $\arg(S(e^{i(\theta+2\pi)}))$ is equal to $\arg(S(e^{i(\theta)}))$.

It can be shown (see e.g. [113, p. 497–498]) that an absolutely summable sequence $s(k)$ and its complex cepstrum are implicitly related as follows:

$$s(k) = \frac{1}{k} \sum_{n=-\infty}^{\infty} n\bar{c}_s(n)s(k-n), \; k \neq 0 \; . \tag{5.7}$$

Based on this expression, the cepstral coefficients of an ARMA process can be obtained from the Markov parameters of the corresponding model and vice versa (see Section 5.5).

5.4 The cepstrum of an ARMA process

Throughout this thesis at different occasions, we have drawn the attention to the properties of SISO ARMA processes (Section 3.2.5, Section 3.5.1, Example 4.14). The examples in this Chapter were also based on them. In the next chapter we will define a norm (and associated metric) for these processes which is based on the cepstrum. Therefore, we discuss in this section the properties of the power cepstrum of ARMA processes. As we will see in Proposition 5.1, the power cepstrum of an ARMA process can be expressed in terms of the input noise variance σ^2 and the poles and zeros of the model. The same is possible for the complex cepstrum of the impulse response of the ARMA model. It does not come as a surprise that these two cepstra are strongly related. How they are related, is given in Corollary 5.2.
Let us first recall that we have defined ARMA models without gain in the transfer function. The gain in the process is modeled by the input variance σ^2 (see also Remark 3.6).

Proposition 5.1. The power cepstrum of a stable and minimum phase ARMA process [113, p. 502]
Let $\{y(k)\}$ be the output of the stable and minimum phase ARMA(p,q) model with poles $\alpha_1, \ldots, \alpha_p$ and zeros $\beta_1, \ldots, \beta_q$ that is driven by white noise with variance σ^2. Then its power cepstrum is equal to

$$c_y(k) = \begin{cases} \log \sigma^2 & k = 0 \; , \\ \displaystyle\sum_{i=1}^{p} \frac{\alpha_i^{|k|}}{|k|} - \sum_{i=1}^{q} \frac{\beta_i^{|k|}}{|k|} & k \neq 0 \; . \end{cases} \tag{5.8}$$

Proof.
The spectral density of the ARMA model can be written as

$$\Phi(e^{i\theta}) = \sigma^2 \frac{\prod_{i=1}^{q} |e^{i\theta} - \beta_i|^2}{\prod_{i=1}^{p} |e^{i\theta} - \alpha_i|^2} = \frac{\prod_{i=1}^{q} |1 - \beta_i e^{-i\theta}|^2}{\prod_{i=1}^{p} |1 - \alpha_i e^{-i\theta}|^2} \; .$$

The logarithm of $\Phi(e^{i\theta})$ is then

$$\log\Phi(e^{i\theta}) = \log(\sigma^2) + \sum_{i=1}^{q}\left(\log(1-\beta_i e^{-i\theta}) + \log(1-\bar{\beta}_i e^{i\theta})\right) - \sum_{i=1}^{p}\left(\log(1-\alpha_i e^{-i\theta}) + \log(1-\bar{\alpha}_i e^{i\theta})\right) \ .$$

Since the poles and zeros lie inside the unit circle, we can apply the power-series expansion $\log(1-x) = -\sum_{k=1}^{\infty}\frac{x^k}{k}$ for $|x| < 1$:

$$\log\Phi(e^{i\theta}) = \log(\sigma^2) + \sum_{i=1}^{p}\left(\sum_{k=1}^{\infty}\frac{\alpha_i^k}{k}e^{-ik\theta} + \sum_{k=1}^{\infty}\frac{\bar{\alpha}_i{}^k}{k}e^{ik\theta}\right) - \sum_{i=1}^{q}\left(\sum_{k=1}^{\infty}\frac{\beta_i^k}{k}e^{-ik\theta} + \sum_{k=1}^{\infty}\frac{\bar{\beta}_i^k}{k}e^{ik\theta}\right) \ . \quad (5.9)$$

On the other hand, the logarithm of the spectrum is also the Fourier transform of the power cepstrum of $\{y(k)\}$:

$$\log\Phi(e^{i\theta}) = \sum_{k=-\infty}^{\infty} c_y(k)e^{-ik\theta} \ . \quad (5.10)$$

By equating the terms in (5.9) and (5.10), we obtain

$$c_y(k) = \begin{cases} \log\sigma^2 & k = 0 \ , \\ \displaystyle\sum_{i=1}^{p}\frac{\alpha_i^k}{k} - \sum_{i=1}^{q}\frac{\beta_i^k}{k} & k > 0 \ , \\ \displaystyle\sum_{i=1}^{q}\frac{\bar{\beta}_i^{-k}}{k} - \sum_{i=1}^{p}\frac{\bar{\alpha}_i^{-k}}{k} & k < 0 \ , \end{cases}$$

and since the poles and zeros occur in complex conjugate pairs,

$$c_y(k) = \begin{cases} \log\sigma^2 & k = 0 \ , \\ \displaystyle\sum_{i=1}^{p}\frac{\alpha_i^{|k|}}{|k|} - \sum_{i=1}^{q}\frac{\beta_i^{|k|}}{|k|} & k \neq 0 \ . \end{cases}$$

□

Note that $c_y(0) = 0$ if the input variance σ^2 is equal to 1.
In Example 5.1 we have seen that the power cepstrum of the ARMA process $\{y(k)\}$ decays faster than its autocovariance function (Figure 5.4). This is partly explained by formula (5.8). While the kth point in the autocovariance

of an ARMA process is a linear combination of the poles to the power $k-1$ (this can be deduced from Equation (3.11)), the kth cepstral coefficient is attenuated by an extra factor $\frac{1}{k}$. That the decay of the cepstrum is not always faster than the decay of the autocovariance function is shown in the following example.

Example 5.4. Autocovariance function and cepstrum of an MA model
We consider the 4th order MA model with transfer function

$$\frac{z^4 + 0.3586z^3 - 0.1994z^2 - 0.8650z - 0.0171}{z^4} .$$

In Figure 5.10 the autocovariance function $R(\tau)$ is given in full line, while the cepstrum of the model $c(k)$ is depicted in dash-dot line. The cepstrum keeps oscillating, while the autocovariance function is equal to zero from $\tau = 5$ onwards. ◊

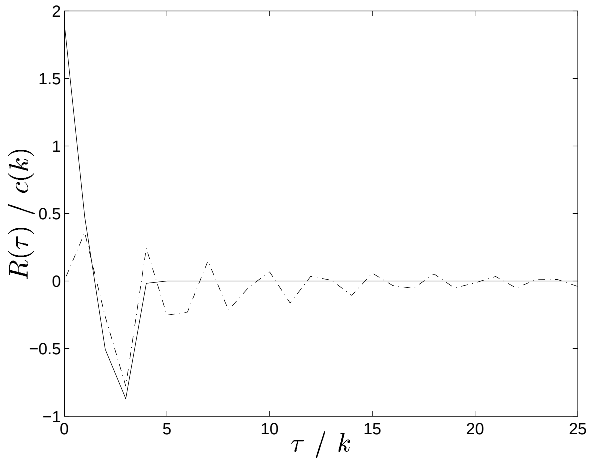


Figure 5.10: The autocovariance (full) and cepstrum (dash-dot) of a 4th order moving average model.

The power cepstrum of the ARMA process $\{y(k)\}$ is related to the complex cepstrum of the impulse response. This result will be needed in Chapter 6.

Corollary 5.2. The power cepstrum of an ARMA process and the complex cepstrum of the model's impulse response
For $k > 0$, the impulse response's complex cepstrum, $\bar{c}_h(k)$, is equal to the power cepstrum $c_y(k)$ of the ARMA process. The coefficients of $\bar{c}_h(k)$ for $k \leq 0$ are zero:

$$\bar{c}_h(k) = \begin{cases} c_y(k) & k > 0 \ , \\ 0 & k \leq 0 \ . \end{cases} \tag{5.11}$$

Proof.
The Z-transform of the impulse response $h(k)$ is the transfer function of the ARMA model $H(z) = \frac{\prod_{i=1}^{q}(1-\beta_i z^{-1})}{\prod_{i=1}^{p}(1-\alpha_i z^{-1})}$. In an analogous way as the proof of Proposition 5.1 one obtains

$$\bar{c}_h(k) = \begin{cases} \sum_{i=1}^{p} \frac{\alpha_i^k}{k} - \sum_{i=1}^{q} \frac{\beta_i^k}{k} & k > 0 \ , \\ 0 & k \leq 0 \ . \end{cases}$$

□

Remark 5.4. The complex cepstrum of an ARMA model
Henceforth we will call the complex cepstrum of the impulse response of an ARMA model briefly the complex cepstrum of the model. ○

Remark 5.5. The complex cepstrum of the inverse model
Let $\bar{c}_H(k)$ be the complex cepstrum of the ARMA model with transfer function $H(z)$. Then, the complex cepstrum of the inverse model, denoted by $\bar{c}_{H^{-1}}(k)$ satisfies

$$\bar{c}_{H^{-1}}(k) = -\bar{c}_H(k) \ .$$

This is easy to see as the poles become zeros and vice versa.
The same holds true for the power cepstra of the output processes of the models, if the noise variance of the input of both models is equal. ○

5.5 System identification based on the cepstrum

Until now, we have worked with the cepstrum as a tool for signal processing, because this is the area where it originated. In this way we could indicate how these ideas emerged. The cepstrum has, however, also made its entry in the system identification community. In particular, methods have been developed for the identification of a linear model of a scalar stationary stochastic process, based on (an estimate of) the power cepstrum of that process. In this section we give a brief overview of two possible methods. In Section 5.5.1 we show how the Markov parameters (i.e. the impulse response sequence) can be computed from the cepstral coefficients. The Markov parameters can then be used to obtain a linear model, as explained in Section 3.3.1. Another approach, called cepstral prediction [111], is explained in Section 5.5.2. Using this method, one obtains the poles and zeros of the ARMA model directly from the cepstrum, without the detour via the Markov parameters.
Byrnes et al. [23, 22] and Enqvist [46, 47] take another approach to the incorporation of cepstral coefficients in system identification. They start from the rational covariance extension problem and bring in the cepstral coefficients in two ways. The first approach is inspired by the maximum entropy model and the second consists in simultaneously interpolating a window of covariance estimates and a window of cepstral coefficients.

5.5.1 Identification of an ARMA model from the cepstral coefficients via the Markov parameters

Proposition 5.1 shows how the power cepstrum of an ARMA process can be computed if the input variance and the poles and zeros of the model are known. It can also be obtained from the input variance and the model's Markov parameters (the impulse response). Conversely, the Markov parameters and the input variance can be derived from the power cepstrum. Indeed, applying (5.7) to the impulse response $h(k)$ of a stable and minimum phase ARMA model, leads to the following recursion:

$$h(k) = \frac{1}{k} \sum_{n=1}^{k} n\bar{c}_h(n)h(k-n) \text{ for } k > 0 \ . \tag{5.12}$$

Since $h(0) = 1$ for the models that we consider (see Section 3.2.5), the expression in (5.12) gives a way to compute the Markov parameters $h(1), \dots, h(N)$ from the complex cepstral coefficients $\bar{c}_h(1), \dots, \bar{c}_h(N)$. Note that in an identification problem we do not have the complex cepstrum $\bar{c}_h(k)$ of the impulse response at our disposal, but an estimate of the power cepstrum $c_y(k)$ of the observed ARMA process. However, from Corollary 5.2 we know that $\bar{c}_h(1) = c_y(1), \dots, \bar{c}_h(N) = c_y(N)$. Consequently, the Markov parameters can be determined from the power cepstrum. Furthermore, the input variance is immediately obtained as $\sigma^2 = e^{c_y(0)}$ (see Proposition 5.1).
Having obtained the Markov parameters, the next step is given by the algorithms designed for the deterministic identification problem, where the model parameters are computed from the Markov parameters, e.g. the algorithm described in Section 3.3.

Example 5.5. Identification of an ARMA model from the power cepstrum estimate
We have generated a time series $y(k)$ by sending a sequence of 50 000 samples of white noise with variance 1 through the ARMA(4, 4) model of Example 5.1. The power cepstrum is estimated via the periodogram, as shown in Figure 5.5 from which the first 100 Markov parameters are computed as in (5.12). Via the deterministic realization algorithm of Kung, which is described in Section 3.3, we obtain an ARMA(4, 4) model. This procedure is repeated 100 times and the location of the resulting poles and zeros of the models is shown in Figure 5.11. ◊

5.5.2 Cepstral prediction

Let the transfer function of an ARMA(n, n) model with poles α_i $(i = 1, \dots, n)$ and zeros β_i $(i = 1, \dots, n)$ be denoted by $H(z) = \frac{b(z)}{a(z)}$ The impulse response

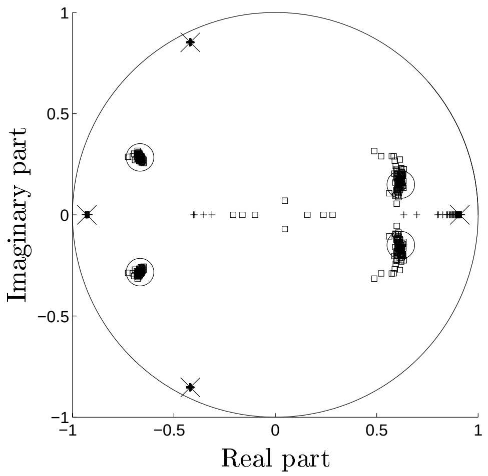


Figure 5.11: The poles ($\times$) and zeros (in the center of the big circles) of the ARMA model used to generate the process $\{y(k)\}$ and the poles ($+$) and zeros ($\square$) of the identified ARMA$(4, 4)$ models.

$h(k)$ of the model can then be written as

$$h(k) = \sum_{i=1}^{n} c_i \alpha_i^k \qquad k > 0 \ ,$$

and also its autocovariance function is a weighted sum of damped exponentials, determined by the poles:

$$R(\tau) = \sum_{i=1}^{n} d_i \alpha_i^\tau \qquad \tau > 0 \ ,$$

where c_i and d_i, $i = 1, \ldots, n$ are weighting constants. From Proposition 5.1 we know that the power cepstrum of the model is equal to

$$c_y(k) = \frac{1}{k} \sum_{i=1}^{n} (\alpha_i^k - \beta_i^k) \text{ for } k > 0 \ ,$$

and thus can the weighted sequence $kc_y(k)$ $(k > 0)$ be viewed as the impulse response or the autocovariance function of an ARMA$(2n, 2n)$ model with poles that correspond to the poles and zeros of the original nth order model. This can also be seen from the Z-transform of $kc_y(k) = k\bar{c}_h(k)$ $(k > 0)$, namely $-z\frac{d}{dz} \log H(z)$, which is a rational function with denominator $a(z)b(z)$.
Given a finite time series $y(k)$ from which the power cepstrum $c_y(k)$ is estimated, a deterministic identification algorithm, or an efficient covariance-based AR identification algorithm applied to $kc_y(k)$ produces a model with poles that

are estimates for the poles and zeros of the original model. However, classifying which pole of the model is a pole and which is a zero of the original model is not always easy. Two methods for this classification problem are given in [111]. Even for data originating from models with an irrational transfer function of the form

$$H(z) = \prod_{i=1}^{p} \frac{1}{(1-\alpha_i z^{-1})^{r_i}} ,$$

where $r_i \in \mathbb{R}$, which are sometimes called fractional models, the sequence $kc_y(k) = k\bar{c}_h(k)$ $(k > 0)$ has a rational Z-transform:

$$-z\frac{d}{dz}\log H(z) = \sum_{i=1}^{p} \frac{r_i \alpha_i z^{-1}}{1-\alpha_i z^{-1}} .$$

Consequently, the same identification technique as described above could be used to find fractional models from cepstral data.

Remark 5.6. Multiplying with k
Although the variance on the estimates of the cepstral coefficients decreases with k, which is shown in [78], one should take care not to amplify numerical and estimation errors when multiplying $c(k)$ by k for large k's. This is illustrated in Figure 5.12, where we consider the spectrum, the cepstrum and the weighted cepstrum of the ARMA model used in the Examples 5.1, 5.3 and 5.5. The power cepstrum of the ARMA process (input is white noise with variance 1) is estimated in two ways. A parametric estimation of the spectrum is obtained by identifying a 10th order AR model. The true spectrum (full line) of the ARMA process and the parametric estimate (dash-dot line) are given in picture A of Figure 5.12. A non-parametric estimate, namely the periodogram without smoothing, which we have computed with the FFT algorithm, is shown in picture B. In picture C, the true power cepstrum (full line), its parametric estimate (dash-dot) and non-parametric estimate (dotted) are depicted. It can be seen that both estimation methods give a good power cepstrum. However, by linearly weighing the estimated cepstra, small differences are amplified, as can be seen in picture D. The parametric estimate (dash-dot) still behaves well, while the non-parametric estimate (dotted) becomes unstable. ○

5.6 Conclusions

Via the motivating example of echo detection and echo removal, we have introduced the cepstrum. The power cepstrum as well as the complex cepstrum were defined, together with their properties that are useful for the derivations in Chapter 6. Furthermore, we have briefly indicated how the cepstrum can be used to identify a linear model. Especially the fact that the zeros of the model can be determined more easily than from the autocovariances, seems promising.

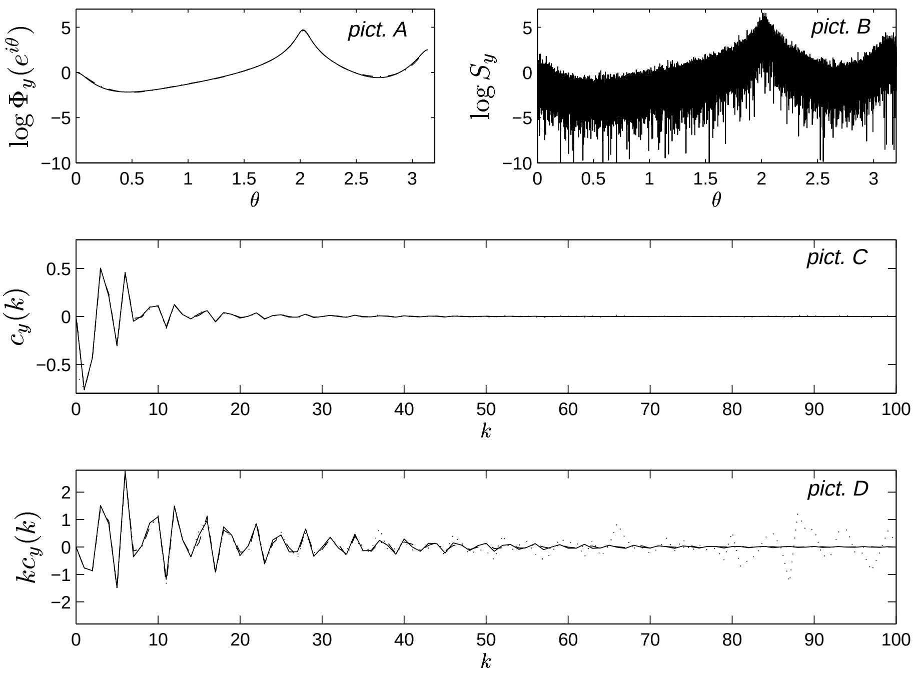


Figure 5.12: The logarithm of the spectrum, power cepstrum and weighted power cepstrum of an ARMA(4, 4) process, estimated in a parametric (via a 10th order AR model) and non-parametric (via the periodogram) way. In picture A the true spectrum (full line) and its parametric estimate (dash-dot) are given. The periodogram is shown in picture B. The true power cepstrum (full line) and its two estimates (parametric in dash-dot, non-parametric in dotted line) are depicted in picture C and in picture D their weighted versions are drawn.