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Lecture 2

- Existence of Solutions
- Uniqueness by Grönwall's Inequality
- Transition Matrix properties
- Adjoint Equation

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General Solution

For continuous $A(t)$, the equation

$$\dot{x}(t) \;=\; A(t)x(t), \quad x(t_0) = x^0$$

has the unique solution

$$x(t) \;=\; \Phi(t, t_0)x^0$$

where $\Phi(t, t_0)$ is the *transition matrix.*

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Transition Matrix

$$\Phi(t,t_0) = I + \int_{t_0}^{t} A(\sigma_1)d\sigma_1$$
$$+ \int_{t_0}^{t} A(\sigma_1)\int_{t_0}^{\sigma_1} A(\sigma_2)d\sigma_2 d\sigma_1$$
$$+ \int_{t_0}^{t} A(\sigma_1)\int_{t_0}^{\sigma_1} A(\sigma_2)\int_{t_0}^{\sigma_2} A(\sigma_3)d\sigma_3 d\sigma_2 d\sigma_1$$
$$\ldots$$

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Example: Time-invariant System

For

$$\dot{x} \;=\; Ax(t), \quad x(t_0) = x^0$$

the transition matrix is

$$\Phi(t,t_0) = I + \int_{t_0}^{t} Ad\sigma_1 + \int_{t_0}^{t} A\int_{t_0}^{\sigma_1} Ad\sigma_2 d\sigma_1 + \cdots$$
$$= I + A(t-t_0) + A^2\frac{(t-t_0)^2}{2} + A^3\frac{(t-t_0)^3}{6} + \cdots$$
$$= e^{A(t-t_0)}$$

so the solution is

$$x(t) \;=\; e^{A(t-t_0)}x^0$$

Example: Scalar Time-variation

Consider

$$\dot{x} = Aa(t)x(t)$$

The transition matrix is

$$\begin{aligned}&\Phi(t,t_0)\\ &= I + A\int_{t_0}^{t} a(\sigma_1)d\sigma_1 + A^2\int_{t_0}^{t} a(\sigma_1)\int_{t_0}^{\sigma_1} a(\sigma_2)d\sigma_2 d\sigma_1 + \cdots\\ &= \sum_{k=0}^{\infty}\frac{1}{k!}A^k\left[\int_{t_0}^{t} a(\sigma)d\sigma\right]^k = \exp\left(A\int_{t_0}^{t} a(\sigma)d\sigma\right)\end{aligned}$$

Second equality is nontrivial.

(Recall Two Tank Example)

General Criterion

If

$$A(t)\int_{t_0}^{t} A(\sigma)d\sigma = \int_{t_0}^{t} A(\sigma)d\sigma A(t)$$

then

$$\Phi(t,t_0) = \exp\left\{\int_{t_0}^{t} A(\sigma)d\sigma\right\}$$

Existence Proof on $[t_0, t_0 + T]$

To solve:

$$\dot{x} = A(t)x(t), \quad x(t_0) = x^0$$

Recursive approximations:

$$\begin{aligned}x_0(t) &= x^0\\ x_1(t) &= x^0 + \int_0^t A(\sigma)x_0(\sigma)d\sigma\\ x_2(t) &= x^0 + \int_0^t A(\sigma)x_1(\sigma)d\sigma\\ &\vdots\end{aligned}$$

Existence Proof Continued

Explicit form:

$$\begin{aligned}&x_k(t) = x^0 + \int_0^t A(\sigma_1)x^0 d\sigma_1\\ &+\cdots+\\ &+\int_0^t A(\sigma_1)\int_0^{\sigma_1} A(\sigma_2)\ldots\int_0^{\sigma_{k-1}} A(\sigma_k)x^0 d\sigma_k\ldots d\sigma_1\end{aligned}$$

Existence Proof Continued

Let

$$\alpha = \max_{t_0 \le t \le t_0+T} \|A(t)\|$$

The maximum exists for continous $A(t)$. Then

$$\begin{aligned} |x_1(t) - x_0(t)| &= \left| \int_{t_0}^{t} A(\sigma) x^0 d\sigma \right| \\ &\le \int_{t_0}^{t} \alpha |x^0| d\sigma \\ &\le (t - t_0)\alpha |x^0| \end{aligned}$$

Existence Proof Continued

$$\begin{aligned} |x_2(t) - x_1(t)| &= \left| \int_{t_0}^{t} A(\sigma)(x_1(\sigma) - x_0(\sigma)) d\sigma \right| \\ &\le \int_{t_0}^{t} \alpha |x_1(\sigma) - x_0(\sigma)| d\sigma \\ &\le \alpha \int_{t_0}^{t} (\sigma - t_0)\alpha |x^0| d\sigma \\ &= \alpha^2 (t - t_0)^2 |x^0|/2 \\ &\vdots \\ |x_k(t) - x_{k-1}(t)| &\le \frac{\alpha^k (t - t_0)^k}{k!} |x^0| \end{aligned}$$

Existence Proof Continued

Hence for $t \in [t_0, T + t_0]$

$$\begin{aligned} \sum_{k=1}^{\infty} |x_k(t) - x_{k-1}(t)| &\le \sum_{k=0}^{\infty} \frac{\alpha^k T^k}{k!} |x^0| \\ &= e^{\alpha T} |x^0| \end{aligned}$$

and by Cauchy's criterion, the sequence $x_1, x_2, \ldots$ converges uniformly on $[t_0, T + t_0]$.

Existence Proof Continued

Taking limits in

$$x_k(t) = x^0 + \int_{t_0}^{t} A(\sigma) x_{k-1}(\sigma) d\sigma$$

gives

$$\begin{aligned} x &= x^0 + \int_{t_0}^{t} A(\sigma) x(\sigma) d\sigma \\ \dot{x}(t) &= A(t)x(t), \quad x(t_0) = x^0 \end{aligned}$$

Example: Non-uniqueness

Let

$$A(t) = \begin{cases} 0 & \text{for } t \le t_0 \\ 1/(t-t_0) & \text{for } t > t_0 \end{cases}$$

Then the equation

$$\dot{x}(t) = A(t)x(t) \qquad\qquad x(0) = 0$$

has two solutions:

$$x \equiv 0 \text{ and } x(t) = \begin{cases} 0 & \text{for } t \le t_0 \\ t - t_0 & \text{for } t > t_0 \end{cases}$$

The Grönwall Inequality

Let $\phi(t)$ and $\nu(t) \ge 0$ be continuous and

$$\phi(t) \le \psi + \int_{t_0}^{t} \nu(\sigma)\phi(\sigma)d\sigma, \quad t \ge t_0$$

for some constant ψ. Then

$$\phi(t) \le \psi e^{\int_{t_0}^{t} \nu(\sigma)d\sigma}, \quad t \ge t_0$$

Proof of Grönwall

Let

$$r(t) = \psi + \int_{t_0}^{t} \nu(\sigma)\phi(\sigma)d\sigma$$

Then

$$\begin{aligned} \dot{r}(t) &= \nu(t)\phi(t) \le \nu(t)r(t) \\ [\dot{r}(t) - \nu(t)r(t)]e^{-\int_{t_0}^{t} \nu(\sigma)d\sigma} &\le 0 \\ \frac{d}{dt}\left[r(t)e^{-\int_{t_0}^{t} \nu(\sigma)d\sigma}\right] &\le 0 \\ r(t)e^{-\int_{t_0}^{t} \nu(\sigma)d\sigma} &\le r(t_0) = \psi \\ \phi(t) \le r(t) &\le \psi e^{\int_{t_0}^{t} \nu(\sigma)d\sigma} \end{aligned}$$

Uniqueness of Solutions

Let

$$\begin{aligned} \dot{x}_a &= A(t)x_a(t), \quad x_a(t_0) = x^0 \\ \dot{x}_b &= A(t)x_b(t), \quad x_b(t_0) = x^0 \\ z(t) &= x_a(t) - x_b(t) \end{aligned}$$

Then

$$\begin{aligned} \dot{z}(t) &= A(t)z(t), \quad z(t_0) = 0 \\ |z(t)| &= \left|\int_0^t A(\sigma)z(\sigma)d\sigma\right| \le \int_0^t \|A(\sigma)\| \cdot |z(\sigma)|d\sigma \end{aligned}$$

and by Grönwall's inequality

$$|z(t)| \equiv 0$$

Characterization of $\Phi(t,t_0)$

The unique solution of the equation

$$\frac{d}{dt}X(t) = A(t)X(t); \quad X(t_0) = I$$

is $X(t) = \Phi(t,t_0)$.

Proof. Let $x(t) = X(t)x^0$. Then

$$\dot{x}(t) = \frac{d}{dt}X(t)x^0 = A(t)X(t)x^0 = A(t)x(t)$$

so

$$x(t) = \Phi(t,t_0)x^0$$

Hence $\Phi(t,t_0)x^0 = X(t)x^0$ for every x^0, so

$$\Phi(t,t_0) = X(t)$$

Input-driven System

The equation

$$\begin{aligned}\dot{x}(t) &= A(t)x(t) + B(t)u(t)\\ x(t_0) &= x^0\end{aligned}$$

has the unique solution

$$x(t) = \Phi(t,t_0)x^0 + \int_{t_0}^{t} \Phi(t,\sigma)B(\sigma)u(\sigma)d\sigma$$

Proof: Differentiate!

Example

$$\begin{aligned}
\dot{x}(t) &= \begin{bmatrix}1 & \cos t\\ 0 & 0\end{bmatrix} x(t)\\
x_2(t) &\equiv x_2^0\\
\dot{x}_1(t) &= x_1(t) + x_2^0 \cos t\\
x_1(t) &= e^t x_1^0 + \int_0^t e^{t-\sigma}\cos\sigma x_2^0 d\sigma = e^t x_1^0 + \frac{1}{2}(e^t + \sin t - \cos t)x_2^0\\
\Phi(t,0) &= \begin{bmatrix} e^t & \frac{1}{2}(e^t + \sin t - \cos t)\\ 0 & 1\end{bmatrix}\\
\Phi(t,t_0) &= \begin{bmatrix} e^{t-t_0} & \frac{1}{2}(e^{t-t_0} + \sin(t-t_0) - \cos(t-t_0))\\ 0 & 1\end{bmatrix}
\end{aligned}$$

Composition Property

For any t, τ, σ, the transition matrix satisfies

$$\Phi(t,\tau) = \Phi(t,\sigma)\Phi(\sigma,\tau)$$

Proof. Let $R(t) = \Phi(t,\sigma)\Phi(\sigma,\tau)$. Then

$$\begin{aligned}\frac{d}{dt}R(t) &= A(t)R(t)\\ R(\sigma) &= \Phi(\sigma,\tau)\end{aligned}$$

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Theorem by Abel-Jacobi-Liouville

Let $A(t)$ be continuous. Then

$$\det\Phi(t,t_0) \quad = \quad \exp\left(\int_{t_0}^{t} \mathrm{Tr}[A(\sigma)]d\sigma\right)$$

Interpretation: Volume contraction

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Proof of Abel-Jacobi-Liouville

$$\begin{aligned}
\frac{d}{dt}\det\Phi(t,t_0) &= \sum_{i,j}\left(\frac{\partial}{\partial\phi_{ij}}\det\Phi(t,t_0)\right)\dot\phi_{ij}(t,t_0)\\
&= \sum_{i,j} c_{ij}(t,t_0)\dot\phi_{ij}(t,t_0)\\
&= \mathrm{tr}\left(C(t,t_0)^T\dot\Phi(t,t_0)\right)\\
&= \mathrm{tr}\left(\Phi(t,t_0)C(t,t_0)^T A(t)\right)\\
&= \mathrm{tr}\left((\det\Phi(t,t_0)I)A(t)\right)\\
&= (\mathrm{tr}A(t))\det\Phi(t,t_0)
\end{aligned}$$

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Inversion

The transition matrix $\Phi(t,t_0)$ is invertible for any t, t_0 and

$$\Phi(t,t_0)^{-1} \quad = \quad \Phi(t_0,t)$$

Proof. Invertibility by the Abel-Jacobi-Liouville Theorem.

Formula by the composition rule

$$I \quad = \quad \Phi(t,t) = \Phi(t,t_0)\Phi(t_0,t)$$

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Change of Variables

The equation

$$\dot x(t) \quad = \quad A(t)x(t), \quad x(t_0) = x^0$$

with new variables $x(t) = P(t)z(t)$ writes

$$\dot z(t) \quad = \quad \left[P(t)^{-1}A(t)P(t) - P(t)^{-1}\dot P(t)\right] z(t)$$

Proof.

$$APz = Ax = \dot x = \dot P z + P\dot z$$

Adjoint Operator

Let $M : X \to Y$ be a bounded linear operator. The *adjoint operator* $M^* : Y \to X$ is defined by the identity

$$< y, Mx > \quad = \quad < M^* y, x >$$

for $x \in X$, $y \in Y$.

Example: Matrix Adjoint

From the equalities

$$< y, Mx > = y^T M x = (M^T y)^T x = < M^T y, x >$$

we see that the adjoint of a matrix is given by its transpose.

Least Squares Problem I

Minimize $|Mu - v|$ with respect to u.

Solution:

$$\hat{u} \quad = \quad (M^* M)^{-1} M^* v \quad \text{(if } M^* M \text{ invertible)}$$

Property:

$$0 \quad = \quad < Mx, M\hat{u} - v > \text{ for all } x$$

Application: Fewer control signals than objectives

Proof

The definition of $\hat{u}$ gives

$$< Mx, M\hat{u} - v > = < x, M^* M\hat{u} - M^* v > = 0$$

With $x = u - \hat{u}$, it follows that

$$\begin{aligned} |Mu - v|^2 &= < M\hat{u} - v + Mx, M\hat{u} - v + Mx > \\ &= < M\hat{u} - v, M\hat{u} - v > + < Mx, Mx > \\ &= |M\hat{u} - v|^2 + |Mx|^2 \\ &\geq |M\hat{u} - v|^2 \end{aligned}$$

This completes the proof

Least Squares Problem II

Minimize $|u|$ under the constraint $Mu = v$.

Solution:

$$\hat{u} \quad = \quad M^*(MM^*)^{-1} v \quad \text{(if } MM^* \text{ invertible)}$$

Property:

$$0 \quad = \quad < \hat{u}, \hat{u} - u > \text{ for all } u \text{ with } Mu = v$$

Application: Reach certain state with minimal cost

Example: Adjoint Transition Matrix

If $M : \mathbf{R}^n \to \mathbf{L}_2^m[0, \infty)$ is defined by

$$(Mx)(t) \quad = \quad \Phi(t,0)x, \quad x \in \mathbf{R}^n$$

then the adjoint $M^* : \mathbf{L}_2^m[0, \infty) \to \mathbf{R}^n$ is given by

$$M^* y \quad = \quad \int_0^\infty \Phi(t,0)^T y(t) dt$$

Proof:

$$\begin{aligned} < y, Mx > \quad &= \quad \int_0^\infty y(t)^T \Phi(t,0) x dt \\ &= \quad \left(\int_0^\infty \Phi(t,0)^T y(t) dt \right)^T x =< M^* y, x > \end{aligned}$$

Next Week

- Transition Matries for Periodic Systems
- Transition Matries for Discrete Time Systems
- Internal Stability