AUTOMATIC CONTROL

Department of Signals, Sensors & Systems

LINEAR SYSTEM THEORY I

Exam October 14–29, 1999

Instructions: Solutions to all problems should be well motivated. There is a total of 60 points, including 13 from the hand in problems. At least 30 should be reached for passed exam.

The examination time is 36 hours. Computers may be used and books may be consulted. You are encouraged to ask me if anything is questionable or difficult to understand, but you may not use help from each other or in any way discuss the exam with other people.

Feedback: I am grateful for your feedback on the course and would also be happy to have your errata collection for the book.

Examiner: Anders Hansson, 790 7425

Results: Will be available by November 5, 1999

Good Luck

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1. (a) Calculate $\exp(Jt)$, where J is an arbitrary Jordan block of dimension n. (3p)

(b) In the proof of Theorem 14.8 Rugh uses the result that for any square matrix A and with $\alpha_m = \|A\|$ it holds that $A + A^T + 2\alpha_m I$ is a positive semidefinite matrix. Prove this result. (3p)

(c) Consider the linear differential equation

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{aligned}$$

which is controllable. Let it be controlled with an observer-based dynamic output feedback

$$\begin{aligned} \dot{\hat{x}}(t) &= A\hat{x}(t) + Bu(t) + H\left[y(t) - C\hat{x}(t)\right] \\ u(t) &= K\hat{x}(t) + r(t) \end{aligned}$$

Show that the closed loop state equation is not controllable. What is the uncontrollable subspace? (3p)

2. Consider the problem of minimizing

$$J = \frac{1}{2}\int_{t_0}^{t_1} \left[x^T(t)Qx(t) + u^T(t)Ru(t)\right] dt$$

subject to

$$\dot{x}(t) = A(t)x(t) + B(t)u(t), \quad x(t_0) = x_0$$

over u, where it is assumed that $Q = Q^T$ and $R = R^T > 0$. A necessary condition for an optimal u is that it satisfies

$$\begin{aligned} \dot{\lambda}(t) &= -A^T(t)\lambda(t) - Qx(t), \quad \lambda(t_1) = 0 \\ u(t) &= -R^{-1}B^T(t)\lambda(t) \end{aligned}$$

(a) Show that for any optimal u it holds that

$$\begin{bmatrix} \dot{x}(t) \\ \dot{\lambda}(t) \end{bmatrix} = \begin{bmatrix} A(t) & -B(t)R^{-1}B^T(t) \\ -Q & -A^T(t) \end{bmatrix} \begin{bmatrix} x(t) \\ \lambda(t) \end{bmatrix}, \quad \begin{bmatrix} x(t_0) \\ \lambda(t_1) \end{bmatrix} = \begin{bmatrix} x_0 \\ 0 \end{bmatrix}$$

Denote the transition matrix of this linear differential equation by $\Phi(t, \tau)$ and partition it as

$$\Phi(t,\tau) = \begin{bmatrix} \Phi_{11}(t,\tau) & \Phi_{12}(t,\tau) \\ \Phi_{21}(t,\tau) & \Phi_{22}(t,\tau) \end{bmatrix}$$

where the partitioning is comformable with the partitioning in the differential equation. Assume that $\Phi_{11}(t, t_1)$ is invertible for all $t \in [t_0, t_1]$ and show that

$$\lambda(t) = \Phi_{21}(t,t_1)\Phi_{11}^{-1}(t,t_1)x(t)$$

for solutions of the above differential equation. (3p)

(b) Let $S(t) = \Phi_{21}(t,t_1)\Phi_{11}^{-1}(t,t_1)$ and show that $S(t)$ satisfies

$$\dot{S}(t) + S(t)A(t) + A^T(t)S(t) - S(t)B(t)R^{-1}B^T(t)S(t) + Q = 0$$

with $S(t_1) = 0$, and that $S(t) = S^T(t)$. (3p)

(c) Show that $u(t) = -R^{-1}B^T(t)S(t)x(t)$ is the solution to the above optimization problem. (Hints: Develop an expression for the derivative of $x^T(t)S(t)x(t)$ with respect to t and use it to rewrite the loss function J. Then express an arbitrary $u(t)$ as $u(t) = -R^{-1}B^T(t)S(t)x(t) + \tilde{u}(t)$ and show that $\tilde{u}(t) \equiv 0$.) (4p)

3. Consider a flexible servo with two tachometers and two motors as in Exercise 6.9. Let

$$\theta(t) = \begin{bmatrix} x_3(t) \\ x_4(t) \end{bmatrix}$$

(a) After Laplace transformation the system can be described by

$$(Js^2 + Ds + K)\Theta(s) = MU(s)$$

where $U(s)$ and $\Theta(s)$ are the Laplace transforms of u and θ, respectively. Determine the matrices J, D, K and M. (2p)

(b) Calculate the Smith form of the polynomial matrix in (a). (3p)

4. Consider the generalized eigenvalue problem

$$(\lambda E + F)x = 0$$

where

$$E = \begin{bmatrix} 0 & -I & 0 \\ A^T & 0 & 0 \\ B^T & 0 & 0 \end{bmatrix}; \quad F = \begin{bmatrix} 0 & A & B \\ -I & Q_1 & Q_{12} \\ 0 & Q_{12}^T & Q_2 \end{bmatrix}$$

and where $Q_1 = Q_1^T$, $Q_2 = Q_2^T$, and A is $n \times n$ and B is $n \times m$.

(a) Show that if $\lambda = \lambda_0 \neq 0$ is an eigenvalue, then so is $\lambda = 1/\lambda_0$. (2p)

(b) Show that if $\lambda = 0$ is an eigenvalue, then so is $\lambda = \infty$. (2p)

(c) Assume that (A, B) is controllable. Show that there are m infinite eigenvalues with linearly independent eigenvectors. (3p)

5. Consider the discrete-time linear system

$$x(k+1) = Ax(k) + Bu(k)$$
$$y(k) = Cx(k)$$

which is assumed to be exponentially stable, reachable and observable.

(a) Show that the reachability and observability grammians

$$P = \sum_{k=0}^{\infty} A^k B B^T \left(A^T\right)^k$$
$$Q = \sum_{k=0}^{\infty} \left(A^T\right)^k C^T C A^k$$

converge absolutely and that they satisfy

$$APA^T - P = -BB^T$$
$$A^T Q A - Q = -C^T C$$

respectively. (3p)

(b) Define a state transformation $\hat{x}(k) = Tx(k)$. Then

$$\hat{x}(k+1) = \hat{A}\hat{x}(k) + \hat{B}u(k)$$
$$y(k) = \hat{C}\hat{x}(k)$$

where $\hat{A} = TAT^{-1}$, $\hat{B} = TB$, and $\hat{C} = CT^{-1}$. Let

$$\hat{P} = \sum_{k=0}^{\infty} \hat{A}^k \hat{B} \hat{B}^T \left(\hat{A}^T\right)^k$$
$$\hat{Q} = \sum_{k=0}^{\infty} \left(\hat{A}^T\right)^k \hat{C}^T \hat{C} \hat{A}^k$$

and show that there is a T such that $\hat{P} = \hat{Q} = \Sigma$, where Σ is a diagonal matrix. (3p)

(c) Show that $\|\hat{A}\| \leq 1$, if T is choosen such that $\hat{P} = \hat{Q} = \Sigma$, where Σ is a diagonal matrix . (3p)

(d) Choose T such that $\hat{P} = \hat{Q} = \Sigma$, where Σ is a diagonal matrix. Partition $\hat{A}$, $\hat{B}$ and $\hat{C}$ as

$$\widehat{A} = \begin{bmatrix} \widehat{A}_{11} & \widehat{A}_{12} \\ \widehat{A}_{21} & \widehat{A}_{22} \end{bmatrix}, \quad \widehat{B} = \begin{bmatrix} \widehat{B}_1 \\ \widehat{B}_2 \end{bmatrix}$$
$$\widehat{C} = \begin{bmatrix} \widehat{C}_1 & \widehat{C}_2 \end{bmatrix}, \quad \Sigma = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix}$$

and assume that the entries of $\hat{x}$ are sorted so that the diagonal elements of Σ are decreasing. Show that the system

$$\bar{x}(k+1) = \hat{A}_{11}\bar{x}(k) + \hat{B}_1 u(k)$$
$$\bar{y}(k) = \hat{C}_1 \bar{x}(k)$$

is exponentially stable, reachable and observable. (7p)