Institutionen för Reglerteknik
Tekniska Högskolan i Lund

Linear Systems I, Brief Solutions

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Participants:
Anna Delfin
Sven Hedlund
Fredrik Kahl
Michael Lantz
Jonas Svensson

1. • Linearize the fermentation model

$$\frac{d}{dt}\begin{bmatrix} V \\ VX \\ G \end{bmatrix} = f(V, VX, G, F) = \begin{bmatrix} F \\ \mu(G)VX \\ \left[(G_{in} - G)F - q_G(G)VX\right]/V(t) \end{bmatrix}$$

$$\mu(G) = Y_x q_G(G), \quad \mu(G) = \mu_{max}\frac{G}{G + k_s}$$

with $Y_x = 0.5$ (g-cells per g-glucose), $\mu_{max} = 0.65h^{-1}$, $k_s = 0.01g/\ell$, $G_{in} = 500g/\ell$, $V(0) = 2\ell$, $VX(0) = 10g$, $G(0) = k_s$ around the nominal glucose feed $F^\circ(t) = F_0 e^{\mu_0 t}$, $\mu_0 = \mu(k_s)$, $(G_{in} - k_s)F_0 = q_G(k_s)VX(0)$.
Nominal trajectory:
$F^\circ(t) = F_0 e^{\mu_0 t}$,
$V^\circ(t) = V(0) + F_0/\mu_0\left(e^{\mu_0 t} - 1\right)$,
$VX^\circ(t) = VX(0)e^{\mu_0 t}$, and
$G^\circ(t) = G(0)$,
gives

$$A(t) = \frac{\partial f}{\partial x} = \mu_0 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & VX(t)/(2k_s) \\ 0 & -1/[Y_x V(t)] & -VX(t)/[2k_s Y_x V(t)] \end{bmatrix}$$

$$B(t) = \frac{\partial f}{\partial u} = [\,1 \quad 0 \quad G_{in}/V(t)\,]^T$$

• Determine the reachability Gramian between $t_1 = 1$ and $t_2 = 2$.
Hard to get a symbolic expression for $\Phi(t, s)$. Use

$$\frac{d}{dt_2}W_r(t_1, t_2) = B(t_2)B^T(t_2) + A(t_2)W_r(t_1, t_2) + W_r(t_1, t_2)A^T(t_2)$$

and solve for $W_r(t_1, t_2)$ numerically starting with $W_r(t_1, t_1) = 0$. The assumptions actually give that $[VX(t)/Y_x + V(t)G(t)]_{t_1}^{t_2} = [V(t)G_{in}]_{t_1}^{t_2}$ for any $F(t)$.
• Discuss also how the "gain" and "timeconstant" of the glucose subsystem changes with time.
The glucose subsystem is much faster than the cell-growth. We have approximately

$$T\frac{dx_3}{dt} + x_3 = Ku$$
$$T = [2k_s Y_x V(t)]/[\mu_0 VX(t)]$$
$$K = [2k_s Y_x G_{in}]/[\mu_0 VX(t)]$$

i.e. T decreases from $T = 22s$ at $t = 0$ to $T = 4.7s$ at $t = 5h$,
while K decreases from $K = 1.5\frac{g/\ell}{\ell/h}$ at $t = 0$ to $K = 0.3\frac{g/\ell}{\ell/h}$ at $t = 5h$. (10 p)

2. Consider the periodic system

$$\dot{x}(t) = -(\sin t + 2)x(t)$$

with period $T = 2\pi$.

• Determine $\Phi(t,\tau)$ and a periodic Lyapunov transformation $x(t) = P(t)z(t)$ giving a timeinvariant z-system.

Scalar system, thus

$$\Phi(t,\tau) = \exp\{-\int_\tau^t (\sin\sigma + 2)d\sigma\} = \exp\{\cos t - \cos\tau - 2(t-\tau)\}$$

With $R = -2$ in $\exp\{RT\} = \Phi(T,0)$ we have $P(t) = \Phi(t,0)\exp\{-Rt\} = \exp\{\cos t - 1\} \in [e^{-2}, 1]$.

• Would there exist initial conditions such that

$$\dot{x}(t) = -(\sin t + 2)x(t) + u(t)$$

has a periodic solution for $u(t) = \sin t$?

$\exists x(0)$ such that $x(t) = x(t+T)$ for any $f(t) = f(t+T)$, since
$1 \neq \Phi(T,0) = e^{-2T}$

(10 p)

3. An electrical system consists of three circuits, each with a resistor and an inductance in series. Assume also coupling between the inductances. Let the first circuit be connected to a voltage source, and let the other two circuits be closed. Thus the system can be described by

$$(sL + R)I(s) = e_1 U(s)$$

where L is a positive definite symmetric matrix of nonnegative inductances, R is a diagonal matrix of positive resistances, and $e_1^T = [1, 0, 0]$.

• Introduce a realization

$$A = -L^{-1}R, \quad B = L^{-1}e_1$$

• and formulate the PBH-test for controllability.

$$\operatorname{rank}[\lambda I - A, B] = \operatorname{rank}[\lambda L + R, e_1] = 3, \quad \forall\lambda$$

• Discuss intuitive parameter combinations resulting in lack of controllability. Assume for simplicity, $L_{1,1} = 1$, $R_{1,1} = 1$.

$L_{1,2} = 0, L_{1,3} = 0$, means no connection.
$L_{2,3} = 0, L_{2,2} = L_{3,3}, R_{2,2} = R_{3,3}$ means two identical circuits by one control.

• Determine the reachable subspace and its dimension for

$$L = \begin{bmatrix} 1 & 1/2 & 1/2 \\ 1/2 & 2 & 1/2 \\ 1/2 & 1/2 & 3 \end{bmatrix} \qquad R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix} \tag{1}$$

Maple

```
L:=matrix(3,3,[1,1/2,1/2,1/2,2,1/2,1/2,1/2,3]):
R:=matrix(3,3,[1,0,0,0,3,0,0,0,5]):
e1:=matrix(3,1,[1,0,0]):
A:=evalm(-inverse(L)&*R):B:=evalm(inverse(L)&*e1):
contr:=concat(B,A&*B,A&*A&*B):
rank(contr);colspace(contr);
```

gives rank 2 and $\mathcal{R}\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 3/5 \end{bmatrix}$.

• There are actually also some nonintuitive combinations. Try finally to get a general condition in terms of $L_{1,2}$, $L_{1,3}$, $L_{2,3}$, $L_{2,2}$, $L_{3,3}$ and $R_{2,2}$, $R_{3,3}$ (quite hard).

$$\operatorname{rank}[sL+R, e_1] = 2 \Leftrightarrow \operatorname{rank}([sL+R, e_1][2..3, 1..3]) = 1$$

i.e. $sL_{1,2}L_{2,3} = L_{1,3}(sL_{2,2} + R_{2,2})$ and $L_{1,2}(sL_{3,3} + R_{3,3}) = sL_{1,3}L_{2,3}$ i.e.

$$R_{3,3}L_{1,2}\left[L_{1,2}L_{2,3} - L_{1,3}L_{2,2}\right] = R_{2,2}L_{1,3}\left[L_{1,3}L_{2,3} - L_{1,2}L_{3,3}\right]$$

Notice that we still also require that $R > 0$, $L > 0$ and $L_{ij} \geq 0$. (10 p)

4. Assume in the previous example with parameters (1) that $L_{3,3} = 3 + 1/1000$. Assume also zero initial currents. Consider the voltage function $\{u(t),\ t \in [0, \infty]\}$ required to achieve $i(\infty) = i_f$.

• Determine the function u_m with minimal 2-norm, i.e. minimizing $||u||^2 = \int_0^\infty u^2(t)dt$. Show how you may utilize the `lyap`-command in Matlab.

Assume first finite final time t_f, i.e $i(t_f) = Lu_{[0,t_f]} = \int_0^{t_f} \Phi(t_f, \tau)Bu(\tau)d\tau$ giving $(L^*y)(t) = B^T\Phi^T(t_f, t)y$ and $u_m = L^*(LL^*)^{-1}i_f$, where $LL^* = W_r(0, t_f)$. Time-invariance and stability gives that $W_r(0, \infty) = P$ is the solution to the Lyapunov equation $AP + PA^T + BB^T = 0$.

$$P = \begin{bmatrix} 0.5456773589 & -0.05710018995 & -0.03425452776 \\ -0.05710018995 & 0.01241350016 & 0.007446189295 \\ -0.03425452776 & 0.007446189295 & 0.004466567554 \end{bmatrix}$$

$W_r(0, t_f)$ converges quite rapidly to P, so $u_m(t) = B^T e^{A^T(t_f - t)}P^{-1}i_f$ is very close to optimal for some large t_f.

• Which combination of currents i_f requires the maximal and minimal $||u_m||$? From $\lambda_{min}(P) = 1.010^{-10}$ and $\lambda_{max}(P) = 0.55$ and the normalized eigenvectors it follows that $i_f = [0.000009704945279, -0.5143658641, 0.8575708473]^T$ gives $||u_m||^2 = 1/\lambda_{min}$, and $i_f = [-0.9924012894, 0.1055134871, 0.06329758955]^T$ gives $||u_m||^2 = 1/\lambda_{max}$. Notice that the eigenvector corresponding to λ_{min} is almost orthogonal to the reachable subspace in problem 3.

(10 p)

5. A linearization of the quadruple-tank process is given by

$$\dot{x} = \begin{bmatrix} -1/T_1 & 0 & 1/T_3 & 0 \\ 0 & -1/T_2 & 0 & 1/T_4 \\ 0 & 0 & -1/T_3 & 0 \\ 0 & 0 & 0 & -1/T_4 \end{bmatrix} x + \begin{bmatrix} b_1 & 0 \\ 0 & b_2 \\ 0 & b_3 \\ b_4 & 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

with $T = [63, 91, 39, 56]$ and $b = [0.048, 0.035, 0.078, 0.056]$.

• Determine the controller form,

```
T=[63,91,39,56];b=[0.048,0.035,0.078,0.056];
A=diag(-1./T);A(1,3)=1/T(3);A(2,4)=1/T(4);
B=[b(1),0;0,b(2);0,b(3);b(4),0];C=[eye(2) zeros(2)];
M=inv([B(:,1),A*B(:,1),B(:,2),A*B(:,2)]);
P=[M(2,:),M(2,:)*A,M(4,:),M(4,:)*A];
U=[M(2,:)*A*A,M(4,:)*A*A];
A0=diag([1 0 1],1);B0=A0([1,3],:)';
Ac=A0+(B0*U)/P,Bc=B0,Cc=C/P
Ac =
         0    1.0000         0         0
   -0.0002   -0.0287    0.0000    0.0026
         0         0         0    1.0000
   -0.0000   -0.0002   -0.0004   -0.0416
Bc =
     0     0
     1     0
     0     0
     0     1
Cc =
    0.0006    0.0480    0.0019   -0.0000
    0.0010    0.0000    0.0011    0.0350
```

• and a state feedback making the poles twice as fast.
Three alternatives, Kp, $Kc1$, $Kc2$:

```
p=-2./T;Kp = place(A,B,p);
poly(p);Kc1=[0 0 -1 0;ans(5:-1:2)]*P+U
p1=poly(p([1,2]));p2=poly(p([3,4]))
Kc2=[p1(3) p1(2) 0 0;0,0,p2(3),p2(2)]*P+U
```

• Determine also a reduced order observer
with poles at $s = -1/10$ and $s = -1/20$.

$$\dot{z} = \tilde{F}z + \tilde{G}_a u + \tilde{G}_b y, \qquad \hat{x} = \begin{bmatrix} y \\ z + Hy \end{bmatrix}$$

```
F11=A(1:2,1:2);F12=A(1:2,3:4);F21=A(3:4,1:2);F22=A(3:4,3:4);
G1=B(1:2,:);G2=B(3:4,:);H=diag([2.9,1.8]);
Ftilde=F22-H*F12;Gatilde=G2-H*G1;Gbtilde=F21-H*F11+Ftilde*H;
```

• Is the resulting controller, i.e. transfer function from y to u, reasonable?

```
K1=-K(:,1:2);K2=-K(:,3:4);
Ar=Ftilde+Gatilde*K2;Cr=K2;Dr=K1+K2*H;Br=Gbtilde+Gatilde*Dr;
eig(Ar),sysr=ss(Ar,Br,Cr,Dr);sysrtf=tf(sysr)
```

Kp and $Kc1$ give reasonable controllers, while $Kc2$ has very high high-frequency gain. All three controllers have stable poles. (10 p)

6. • Use Rugh's method (Corollary 14.13) to get noninteracting control of the quadrupel tank. Determine the Markov parameters and the relative degrees.

```
G0=C*B,Delta=G0;Omega=C;
K=-Delta\(Omega*A),N=inv(Delta)
sysg=ss(A+B*K,B*N,C,zeros(2));
tf(sysg),eig(A+B*K)
```

$$K = \begin{bmatrix} 0.3307 & 0 & -0.5342 & 0 \\ 0 & 0.3140 & 0 & -0.5102 \end{bmatrix}, \quad N = \begin{bmatrix} 20.8333 & 0 \\ 0 & 28.5714 \end{bmatrix}$$

gives the closed loop system $G_c(s) = I * \frac{1}{s}$ after cancellation of the two closed loop poles $\{-0.0565, 0.0130\}$.

(5 p)

7. In the enclosed very recent paper is calculated among other things the singular values of the controllability matrix for a triple inverted pendulum. Assume that the system is initially at rest, except for a deviation of 5 degrees in the third link. Use the controller (18).

• Check the closed loop eigenvalues

```
B=[0 0 0 0 0.9033 -2.020 1.9195 0.0904]';
A=[zeros(4),eye(4);0 -7.6199 -0.1568 -0.0005 -4.9681 0.0005 -0.0005 0;
0 38.978 -23.9878 -0.0784 11.1101 -0.0046 0.0087 -0.0037;
0 -37.0386 82.7535 -2.0117 -10.5573 0.0087 -0.0234 0.0253;
0 -1.7447 -52.8669 71.9997 -0.4973 -0.0037 0.0253 -0.4028];
K18=[45.5 246.5 -1007 2656 38.8 102.6 28.1 313.7];
eig(A+B*K18)
lamc=[-21.48+6.53*i -6.26+6.01*i -2.48+4.04*i  -1.34+1.78*i ];
lamc=[lamc,conj(lamc)];
K=-place(A,B,lamc)
```

Obtained eigenvalues are actually unstable. There is something wrong with $K18$, but $K = [50.82, 361.7, -704.6, 3129, 57.23, 141.75, 75.34, 357.2]$ gives the desired eigenvalues.

• and determine the square integral of the control signal. Hint: Determine a Lyapunov equation for the integral

$$x_0^T \left(\int_0^\infty e^{(A+BK)^T t} K^T K e^{(A+BK)t} dt \right) x_0$$

```
P=lyap((A+B*K)',K'*K);
zPz=P(4,4)*(5*pi/180)^2
```

• Use Matlab, `c2d`, to sample the system with sampling interval $h = 12$ms.

```
[Phi,Gamma]=c2d(A,B,0.012)
```

• Minimize the control signal norm to reach the origin in 1000 sampling intervals, using the discrete time controllability Gramian $W_c(0,1000)$. One idea would be to find a recursion for $W_c(0,k)$.
Determine u_m minimizing $||u||$ in

$$0 = \Phi(k_f,0)x_0 + R(0,k_f)u_{[0,k_f-1]}$$

With

$$\begin{aligned} x_0 &= Lu = L(0,k_f)u_{[0,k_f-1]} \\ L(0,k_f) &= [\Phi^{-1}(k_f,0)\Gamma, \ldots, \Phi^{-1}(k_f,0)\Phi(k_f,1)\Gamma] = [\Phi^{-k_f}\Gamma, \ldots, \Phi^{-1}\Gamma] \end{aligned}$$

we have

$$u_m = L^*(LL^*)^{-1}x_0, \qquad W_c(0,k_f) = L(0,k_f)L^T(0,k_f)$$

Therefore

$$\begin{aligned} L(0,k_f+1) &= [\Phi^{-1}L(0,k_f),\ \Phi^{-1}\Gamma] \\ W_c(0,k+1) &= \Phi^{-1}W_c(0,k)\Phi^{-T} + \Phi^{-1}\Gamma(\Phi^{-1}\Gamma)^T \end{aligned}$$

The instability and numerical roundoff errors make it hard to use the recursion.

```
PiG=Phi\Gamma;Q=PiG*PiG';Wc=Q;kf=100;
for k=1:kf,Wc=Phi\(Wc/Phi)+Q; end
Wci=inv(Wc);
Wci(4,4)*(5*pi/180)^2
```

(5 p)