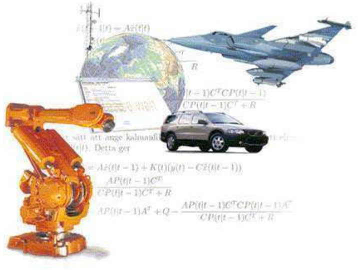
Discretizing stochastic dynamical systems using Lyapunov equations



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Motivation

- **Stochastic dynamical systems** are important in state estimation, system identification and control.
- System models are often **provided in continuous time**, while a major part of the **applied theory** is developed for **discrete-time systems**.
- **Discretization** of continuous-time models is hence fundamental.

Discretization

Continuous-time model

$$\dot{x}(t) = Ax(t) + Bw(t)$$
$$E[w(t)w(\tau)^{\mathsf{T}}] = S\delta(t-\tau)$$

Discrete-time model

$$x_{k+1} = Fx_k + w_k$$
$$E[w_k w_l^{\mathsf{T}}] = Q\delta_{kl}$$

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This gives the relations

$$F = e^{AT_s}, \qquad Q = \int_0^{T_s} e^{A\tau} B S B^\mathsf{T} e^{A^\mathsf{T}\tau} d\tau$$

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Problem

How do we solve this integral in a numerically good manner for arbitrary A, B, S and T_s?

If the system has only integrators, A is nilpotent and $e^{A\tau}$ can be expressed exactly with a finite Taylor series

$$e^{A\tau} = I + A\tau + \cdots + \frac{1}{p!}A^{p-1}\tau^{p-1}.$$

Then, the integral has an analytical solution

$$Q = \int_0^{T_s} e^{A\tau} BSB^\mathsf{T} e^{A^\mathsf{T}\tau} d\tau$$
$$= \sum_{i=0}^{p-1}\sum_{j=0}^{p-1} \frac{T_s^{i+j+1}}{i!j!(i+j+1)} A^i S (A^i)^\mathsf{T}.$$

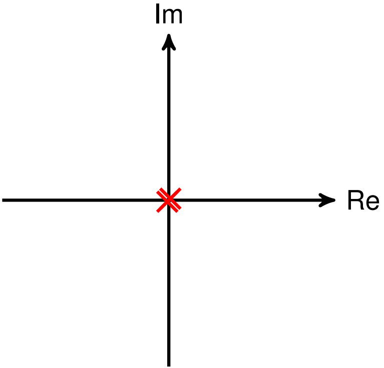

Figure : The poles of the system

Theorem

Theorem - Contribution 1

The solution to the integral

$$Q = \int_0^{T_s} e^{A\tau} B S B^{\mathsf{T}} e^{A^{\mathsf{T}}\tau} d\tau \tag{1}$$

satisfies the Lyapunov equation

$$AQ + QA^{\mathsf{T}} = -BSB^{\mathsf{T}} + e^{AT_s} BSB^{\mathsf{T}} e^{A^{\mathsf{T}} T_s} \tag{2}$$

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Idea

Solve (2) to find solution for (1).

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Solve (2) to find solution for (1).

Lemma *Eq.* (2) *has a unique solution if and only if*

$$\lambda_i(A) + \lambda_j(A) \neq 0 \quad \forall i,j$$

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Solve (2) to find solution for (1).

Lemma *Eq.* (2) *has a unique solution if and only if*

$$\boxed{\lambda_i(A) + \lambda_j(A) \neq 0 \quad \forall i, j}$$

Note: This is not fulfilled if the system has integrators!

If the system has only strictly stable poles, the Lyapunov equation has a unique solution

$$AQ + QA^{\mathsf{T}} = \underbrace{-BSB^{\mathsf{T}} + e^{AT_s}BSB^{\mathsf{T}}e^{A^{\mathsf{T}}T_s}}_{-V},$$

which is identical to the solution of the integral

$$Q = \int_0^{T_s} e^{A\tau}BSB^{\mathsf{T}}e^{A^{\mathsf{T}}\tau}d\tau.$$

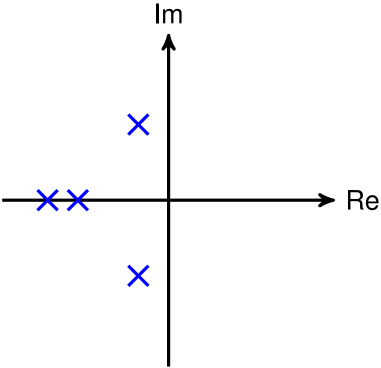
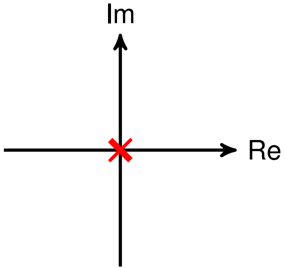
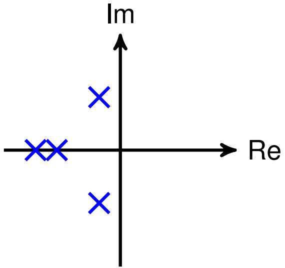


Figure : The poles of the system

Summary so far





$$Q = \sum_{i=0}^{p-1} \sum_{j=0}^{p-1} \frac{T_s^{i+j+1}}{i!j!(i+j+1)} A^i S (A^i)^\mathsf{T}$$

$$AQ + QA^\mathsf{T} = -V$$

How do we find Q if we have **both** integrators **and** strictly stable poles?

Consider a system on the following block triangular form

$$A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$$

where all integrators are collected in A_{22}, i.e.

$$\lambda_i(A_{11}) \neq 0 \quad \forall i$$
$$\lambda_j(A_{22}) = 0 \quad \forall j$$

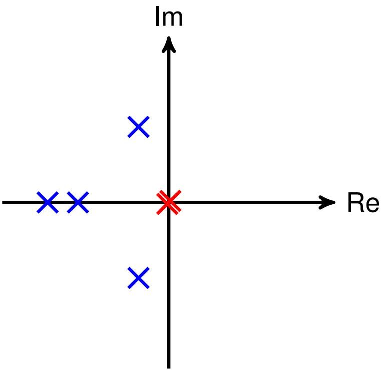


Figure : The poles of the system

The corresponding Lyapunov equation for this system reads

$$\begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^\mathsf{T} & Q_{22} \end{bmatrix} + \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^\mathsf{T} & Q_{22} \end{bmatrix} \begin{bmatrix} A_{11}^\mathsf{T} & 0 \\ A_{12}^\mathsf{T} & A_{22}^\mathsf{T} \end{bmatrix} = - \begin{bmatrix} V_{11} & V_{12} \\ V_{12}^\mathsf{T} & V_{22} \end{bmatrix},$$

which gives

$$\begin{aligned} A_{11}Q_{11} + Q_{11}A_{11}^\mathsf{T} &= -V_{11} - A_{12}Q_{12}^\mathsf{T} - Q_{12}A_{12}^\mathsf{T}, \\ A_{11}Q_{12} + Q_{12}A_{22}^\mathsf{T} &= -V_{12} - A_{12}Q_{22}, \\ A_{22}Q_{12}^\mathsf{T} + Q_{12}^\mathsf{T}A_{11}^\mathsf{T} &= -V_{12}^\mathsf{T} - Q_{22}A_{12}^\mathsf{T}, \\ A_{22}Q_{22} + Q_{22}A_{22}^\mathsf{T} &= -V_{22} \end{aligned}$$

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The corresponding Lyapunov equation for this system reads

$$\begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^T & Q_{22} \end{bmatrix} + \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^T & Q_{22} \end{bmatrix} \begin{bmatrix} A_{11}^T & 0 \\ A_{12}^T & A_{22}^T \end{bmatrix} = - \begin{bmatrix} V_{11} & V_{12} \\ V_{12}^T & V_{22} \end{bmatrix},$$

which gives

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Proposed solution:

- contribution 2

1. Find Q_{22} by using the analytical solution.

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1. Find Q_{22} by using the analytical solution.
2. Find Q_{12} by solving a Sylvester equation.

Systems with integrators + strictly stable poles

The corresponding Lyapunov equation for this system reads

$$\begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^{\mathsf{T}} & Q_{22} \end{bmatrix} + \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^{\mathsf{T}} & Q_{22} \end{bmatrix} \begin{bmatrix} A_{11}^{\mathsf{T}} & 0 \\ A_{12}^{\mathsf{T}} & A_{22}^{\mathsf{T}} \end{bmatrix} = - \begin{bmatrix} V_{11} & V_{12} \\ V_{12}^{\mathsf{T}} & V_{22} \end{bmatrix},$$

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A_{22}Q_{12}^{\mathsf{T}} + Q_{12}^{\mathsf{T}}A_{11}^{\mathsf{T}} &= -V_{12}^{\mathsf{T}} - Q_{22}A_{12}^{\mathsf{T}}, \\
A_{22}Q_{22} + Q_{22}A_{22}^{\mathsf{T}} &= -V_{22}
\end{aligned}$$

Proposed solution:
- contribution 2

1. Find Q_{22} by using the analytical solution.
2. Find Q_{12} by solving a Sylvester equation.
3. Find Q_{11} by solving a Lyapunov equation.

Van Loan's method (standard method in literature)

1. Form the $2n \times 2n$ matrix

$$H = \begin{bmatrix} A & BSB^{\mathsf{T}} \\ 0 & -A^{\mathsf{T}} \end{bmatrix}.$$

2. Compute the matrix exponential

$$e^{HT_s} = \begin{bmatrix} M_{11} & M_{12} \\ 0 & M_{22} \end{bmatrix}.$$

3. Then Q is given as

$$Q = M_{12} M_{11}^{\mathsf{T}}.$$

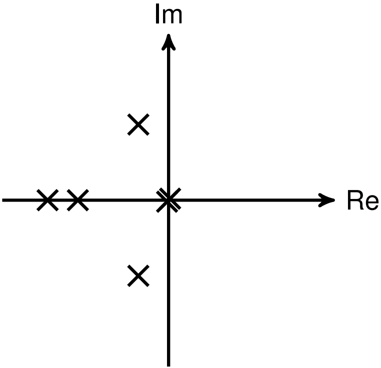


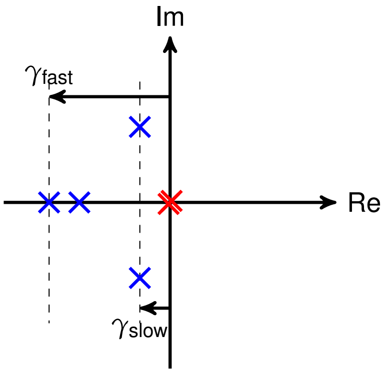
Figure : The poles of the system

Van Loan, C.F. (1978). **Computing integrals involving the matrix exponential**.

IEEE Transactions on Automatic Control, 23(3), 395-404.

Numerical Evaluation

- Marginally stable systems with 4 stable poles and 2 integrators are considered.
- The 2D region $\gamma_{\text{slow}}, \gamma_{\text{fast}} \in [10^{-1}, 10^{1}]$ is divided into 25×25 bins.
- In total 100 systems are randomly generated for each bin.
- Consider the sampling time $T_s = 1$.



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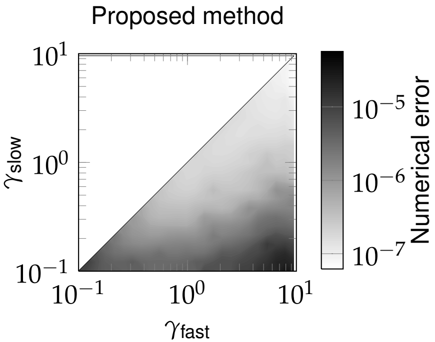
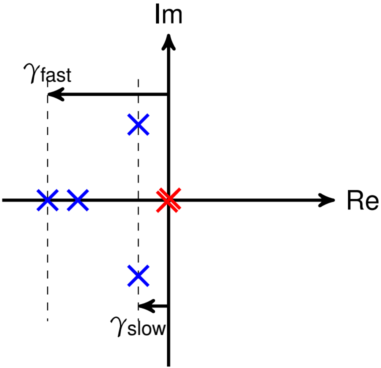


If γ_{slow} is small, the condition is closer to be violated

$$\lambda_i(A_{11}) + \lambda_j(A_{11}) \neq 0 \quad \forall i,j$$



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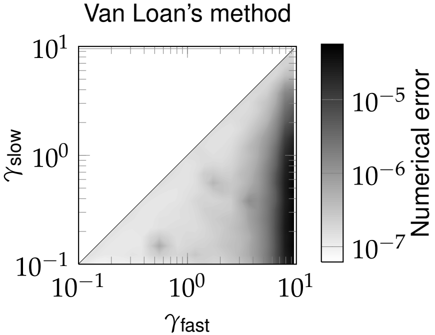
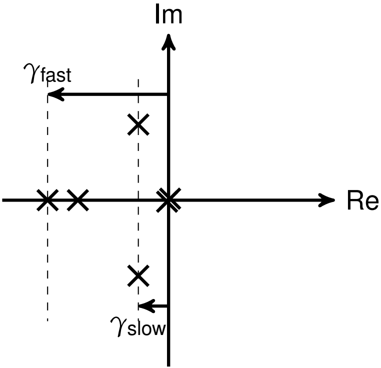


If γ_{fast} is large, the matrix exponential is ill-conditioned

$$\exp\left(\begin{bmatrix} A & BSB^{\mathsf{T}} \\ 0 & -A^{\mathsf{T}} \end{bmatrix} T_s\right)$$



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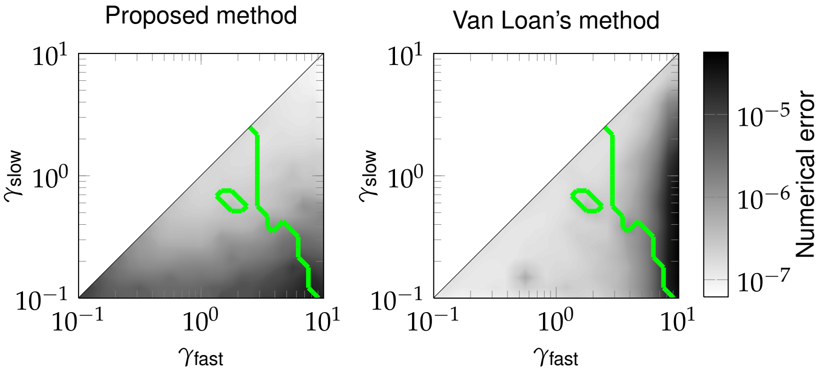

- The proposed method performs better if the slowest pole is fast.
- Van Loan's method performs better if the fastest pole is slow.

Conclusions

- An algorithm for computing an **integral involving the matrix exponential** common in optimal sampling has been proposed.
- The algorithm is **based on a Lyapunov equation** and is justified with a novel theorem.
- Numerical evaluations showed that the proposed algorithm has **advantageous numerical properties** for slow sampling and fast dynamics in comparison with a standard method in the literature.

Analytical solution - Example

Consider a **double integrator** (two zero eigenvalues of A)

$$\dot{x}(t) = \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{A} x(t) + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B} w(t), \;\; E[w(t)w(\tau)] = \gamma\delta(t-\tau).$$

Only zero eigenvalues $\Leftrightarrow$ nilpotent, so

$$e^{A\tau} = I + A\tau = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \tau$$

Q can be computed analytically

$$Q = \int_0^{T_s} e^{A\tau} B S B^{\mathsf{T}} e^{A^{\mathsf{T}}\tau} d\tau = \gamma \begin{bmatrix} \frac{T_s^3}{3} & \frac{T_s^2}{2} \\ \frac{T_s^2}{2} & T_s \end{bmatrix}.$$

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Continuous-time model

$$\dot{x}(t) = Ax(t) + Bw(t)$$
$$E[w(t)w(\tau)^{\mathsf{T}}] = S\delta(t-\tau)$$

Discrete-time model

$$x_{k+1} = F_T x_k + w_k$$
$$E[w_k w_l^{\mathsf{T}}] = Q_T \delta_{kl}$$

...or more strict

$$dx(t) = Ax(t) + Bdw(t)$$
$$E[dw(t)dw(t)^{\mathsf{T}}] = Sdt$$

where $w(t)$ a Brownian motion.